

THETA INVARIANTS OF EUCLIDEAN LATTICES AND INFINITE-DIMENSIONAL HERMITIAN VECTOR BUNDLES OVER ARITHMETIC CURVES

JEAN-BENOÎT BOST

ABSTRACT. In this article, we lay foundations for a theory of infinite dimensional euclidean lattices — and more generally, of infinite dimensional hermitian vector bundles over some “arithmetic curve” $\mathrm{Spec} \mathcal{O}_K$ attached to the ring of integers \mathcal{O}_K of some number field K — with a view towards applications to transcendence theory and Diophantine geometry.

In the first part of this article, we study the properties of the θ -invariant $h_\theta^0(\overline{E})$ attached to some euclidean lattice $\overline{E} := (E, \|\cdot\|)$, defined by the expression

$$h_\theta^0(\overline{E}) := \log \sum_{v \in E} e^{-\pi \|v\|^2},$$

and, more generally, of the θ -invariant attached to some finite rank hermitian vector bundle \overline{E} over an arithmetic curve.

Then we construct categories of infinite dimensional hermitian vector bundles and we show that it is possible to associate generalized θ -invariants to their objects, so that they satisfy suitable subadditivity and limit properties.

CONTENTS

1. Introduction	3
1.1. Pro-euclidean lattices and θ -invariants	3
1.2. Contents	6
1.3. Acknowledgements	8
1.4. Notation and conventions	9
2. Hermitian vector bundles over arithmetic curves	9
2.1. Definitions and basic operations	9
2.2. Direct images. The canonical hermitian line bundle $\overline{\omega}_{\mathcal{O}_K/\mathbb{Z}}$ over $\mathrm{Spec} \mathcal{O}_K$	10
2.3. Arakelov degree	11
2.4. Morphisms and extensions of hermitian vector bundles	12
3. θ -Invariants of hermitian vector bundles over arithmetic curves	17
3.1. The Poisson formula	18
3.2. The θ -invariants h_θ^0 and h_θ^1 and the Poisson-Riemann-Roch formula	19
3.3. Positivity and monotonicity	20
3.4. The functions τ and η	23
3.5. Additivity. The θ -invariants of direct sums of hermitian line bundles over $\mathrm{Spec} \mathbb{Z}$	24
3.6. The theta function $\theta_{\overline{E}}$ and the first minimum $\lambda_1(\overline{E})$	25
3.7. Application to hermitian line bundles	28

3.8. Subadditivity of h_θ^0 and h_θ^1	30
4. Geometry of numbers and θ -invariants	33
4.1. Comparing h_θ^0 and h_{Ar}^0	34
4.2. Banaszczyk's estimates and θ -invariants	36
4.3. Subadditive invariants of euclidean lattices	40
4.4. The asymptotic invariant $\tilde{h}_{\text{Ar}}^0(\overline{E}, t)$	43
4.5. Some consequences of Siegel's mean value theorem	48
5. Countably generated projective modules and linearly compact Tate spaces over Dedekind rings	54
5.1. Countably generated projective A -modules	54
5.2. Linearly compact Tate spaces with countable basis	56
5.3. The duality between CP_A and CTC_A	60
5.4. Strict morphisms, exactness and duality	63
5.5. Examples	70
6. Ind- and pro-hermitian vector bundles over arithmetic curves	74
6.1. Definitions	75
6.2. Hilbertisable ind- and pro-vector bundles	79
6.3. Constructions as inductive and projective limits	80
6.4. Morphisms between ind- and pro-hermitian vector bundles over \mathcal{O}_K	82
6.5. The duality between ind- and pro-hermitian vector bundles	85
6.6. Examples – I. Formal series and holomorphic functions on disks	90
6.7. Examples. II – Injectivity and surjectivity of morphisms of pro-hermitian vector bundles	91
6.8. Examples. III – Subgroups of pre-Hilbert spaces and ind-euclidean lattices	94
7. θ -Invariants of infinite dimensional hermitian vector bundles: definitions and first properties	95
7.1. Limits of θ -invariants	96
7.2. Upper and lower θ -invariants	97
7.3. Basic properties	100
7.4. Examples	104
8. Summable projective systems of hermitian vector bundles and finiteness of θ -invariants	107
8.1. Main theorem	107
8.2. Preliminaries	108
8.3. Summable projective systems of hermitian vector bundles and associated measures	110
8.4. Proof of Theorem 8.3.4 – I. The equality $\overline{h}_\theta^0(\hat{\overline{E}}) = \lim_{i \rightarrow +\infty} h_\theta^0(\overline{E}_i)$	112
8.5. Proof of Theorem 8.3.4 – II. Convergence of measures	114
8.6. A converse theorem	116
8.7. Strongly summable and θ -finite pro-hermitian vector bundles	118
9. Subadditivity properties of the θ -invariants attached to infinite dimensional hermitian vector bundles	121
9.1. Short exact sequences of infinite dimensional hermitian vector bundles	121
9.2. Short exact sequences and θ -invariants of pro-hermitian vector bundles	123

9.3. Short exact sequences and strongly summable pro-hermitian vector bundles	129
Appendix A. Large deviations and Cramér's theorem	133
A.1. Notation and preliminaries	134
A.2. Lanford's inequalities	136
A.3. Cramér's theorem	138
A.4. An extension of Cramér's Theorem concerning general positive measures	140
A.5. Reformulation and complements	143
Appendix B. Non-complete discrete valuation rings and continuity of linear forms on prodiscrete modules	150
B.1. Preliminary: maximal ideals, discrete valuation rings, and completions	150
B.2. Continuity of linear forms on prodiscrete modules	151
Appendix C. Measures on countable sets and their projective limits	152
C.1. Finite measures on countable sets	153
C.2. Finite measures on projective limits of countable sets	154
References	158

1. INTRODUCTION

1.1. Pro-euclidean lattices and θ -invariants. In this article, we lay foundations for a theory of *infinite dimensional euclidean lattices* — and more generally, of infinite dimensional hermitian vector bundles over some “arithmetic curve” $\text{Spec } \mathcal{O}_K$ attached to the ring of integers \mathcal{O}_K of some number field K — with a view towards applications to transcendence theory and Diophantine geometry.

1.1.1. Recall that an *euclidean lattice* is the data

$$\overline{E} := (E, \|\cdot\|)$$

of some free \mathbb{Z} -module of finite rank E and of some euclidean norm $\|\cdot\|$ on the real vector space $E_{\mathbb{R}} := E_{\mathbb{R}}$.

The infinite dimensional generalizations of euclidean lattices we will be mainly interested in are the *pro-euclidean lattices*. They naturally occur as projective limits of countable systems of euclidean lattices

$$\overline{E}_{\bullet} : \overline{E}_0 \xleftarrow{q_0} \overline{E}_1 \xleftarrow{q_1} \dots \xleftarrow{q_{i-1}} \overline{E}_i \xleftarrow{q_i} \overline{E}_{i+1} \xleftarrow{q_{i+1}} \dots$$

Here, for every $i \in \mathbb{N}$, we denote by \overline{E}_i some euclidean lattice $(E_i, \|\cdot\|_i)$ and by q_i a surjective morphism of \mathbb{Z} -modules

$$q_i : E_{i+1} \longrightarrow E_i$$

such that the norm $\|\cdot\|_i$ on $V_{i,\mathbb{R}}$ coincides with the the quotient norm deduced from the norm $\|\cdot\|_{i+1}$ on $E_{i+1,\mathbb{R}}$ by means of the surjective \mathbb{R} -linear map

$$q_{i,\mathbb{R}} := q_i \otimes Id_{\mathbb{R}} : E_{i+1,\mathbb{R}} \longrightarrow E_{i,\mathbb{R}}.$$

A *pro-euclidean lattice* may actually be defined “directly”, without explicit mention of projective systems of euclidean lattices, as a triple

$$\widehat{E} := (\widehat{E}, E_{\mathbb{R}}^{\text{Hilb}}, \|\cdot\|)$$

consisting in the following data:

- (1) an abelian topological group \widehat{E} , isomorphic to \mathbb{Z}^n (for some $n \in \mathbb{N}$) equipped with the discrete topology, or to $\mathbb{Z}^{\mathbb{N}}$ equipped with the product topology of the discrete topology on each factor \mathbb{Z} ;
- (2) a dense real vector subspace $E_{\mathbb{R}}^{\text{Hilb}}$ of the topological real vector space $\widehat{E}_{\mathbb{R}} := \widehat{E} \widehat{\otimes}_{\mathbb{Z}} \mathbb{R}$, defined as the completed tensor product¹ of \widehat{E} by \mathbb{R} ;
- (3) a norm $\|\cdot\|$ on $E_{\mathbb{R}}^{\text{Hilb}}$ that makes $(E_{\mathbb{R}}^{\text{Hilb}}, \|\cdot\|)$ a real Hilbert space; this Hilbert space topology on $E_{\mathbb{R}}^{\text{Hilb}}$ is moreover required to be finer than the topology induced by the topology of $\widehat{E}_{\mathbb{R}}$.

To any projective system \overline{E}_{\bullet} as above, one attaches such data by defining \widehat{E} as the pro-discrete \mathbb{Z} -module

$$\widehat{E} := \varprojlim_i E_i$$

and $(E_{\mathbb{R}}^{\text{Hilb}}, \|\cdot\|)$ as the projective limit, in the category of real normed vector spaces, of the projective system:

$$(E_{0,\mathbb{R}}, \|\cdot\|_0) \xleftarrow{q_{0,\mathbb{R}}} (E_{1,\mathbb{R}}, \|\cdot\|_1) \xleftarrow{q_{1,\mathbb{R}}} \cdots \xleftarrow{q_{i-1,\mathbb{R}}} (E_{i,\mathbb{R}}, \|\cdot\|_i) \xleftarrow{q_{i,\mathbb{R}}} (E_{i+1,\mathbb{R}}, \|\cdot\|_{i+1}) \xleftarrow{q_{i+1,\mathbb{R}}} \cdots$$

Indeed \widehat{E} may be identified with the closed submodule of $\prod_{i \in \mathbb{N}} E_i$ consisting of families $(v_i)_{i \in \mathbb{N}}$ such that

$$(1.1) \quad \text{for any } i \in \mathbb{N}, v_i = q_i(v_{i+1}).$$

Similarly the topological real vector space $\widehat{E}_{\mathbb{R}}$ may be identified with the closed subspace of $\prod_{i \in \mathbb{N}} E_{i,\mathbb{R}}$ consisting of families $(v_i)_{i \in \mathbb{N}}$ satisfying (1.1), and $E_{\mathbb{R}}^{\text{Hilb}}$ with the subspace of $\prod_{i \in \mathbb{N}} E_{i,\mathbb{R}}$ consisting of families $(v_i)_{i \in \mathbb{N}}$ such that, besides (1.1), the following holds:

$$\|(v_i)_{i \in \mathbb{N}}\| := \lim_{i \rightarrow +\infty} \|v_i\|_i < +\infty.$$

(For any $(v_i)_{i \in \mathbb{N}}$ in $\prod_{i \in \mathbb{N}} E_{i,\mathbb{R}}$ satisfying (1.1), the limit $\lim_{i \rightarrow +\infty} \|v_i\|_i$ exists in $[0, +\infty]$ since $(\|v_i\|_i)_{i \in \mathbb{N}}$ is a non-decreasing sequence in \mathbb{R}_+ .)

These descriptions of \widehat{E} , $\widehat{E}_{\mathbb{R}}$ and $E_{\mathbb{R}}^{\text{Hilb}}$ make clear the inclusions

$$\widehat{E} \hookrightarrow \widehat{E}_{\mathbb{R}} \hookrightarrow E_{\mathbb{R}}^{\text{Hilb}},$$

and shows that the triple $(\widehat{E}, E_{\mathbb{R}}^{\text{Hilb}}, \|\cdot\|)$ actually satisfies the conditions (1)–(3) above and defines a pro-euclidean lattice that we shall denote $\varprojlim_i \overline{E}_i$.

1.1.2. It turns out that meaningful invariants may be attached to the so-defined pro-euclidean lattices. In this article, we shall focus on their θ -invariants.

Recall that, in the classical analogy between number fields and function fields, an euclidean lattice $\overline{E} := (E, \|\cdot\|)$ appears as the counterpart of a vector bundle on some smooth projective curve C over some base field k . Then the arithmetic counterpart of the dimension

$$(1.2) \quad h^0(C, V) := \dim_k \Gamma(C, V)$$

of the space of sections of V is the non-negative real number

$$(1.3) \quad h_{\theta}^0(\overline{E}) := \log \sum_{v \in E} e^{-\pi \|v\|^2}.$$

¹Any isomorphism of topological groups $\phi : \widehat{E} \xrightarrow{\sim} \mathbb{Z}^N$ of \widehat{E} by \mathbb{R} , with $N \in \mathbb{N}$ or $N = \mathbb{N}$, induces an isomorphism $\phi_{\mathbb{R}} := \phi \widehat{\otimes}_{\mathbb{Z}} : \widehat{E}_{\mathbb{R}} \rightarrow \mathbb{R}^N$ of topological real vector spaces, where the topology on \mathbb{R}^N is defined as the product of the usual topology on each factor \mathbb{R} .

The correspondence between the invariants (1.2) and (1.3), in the geometric and arithmetic situations, goes back at least² to the work of F. K. Schmidt.

Indeed, Schmidt's proof in [Sch31] of the functional equation for the zeta function of some function field $k(C)$ attached to some smooth projective curve C over a finite field k , crucially relies on the Riemann-Roch formula for line bundles over C . Compared with the earlier proof by Hecke ([Hec17]) of the functional equation for the Dedekind zeta function of an arbitrary number field K — where the Poisson formula for euclidean lattices of rank $[K : \mathbb{Q}]$ plays a key role — Schmidt's arguments make clear the correspondence between (1.2) and (1.3), and the fact that, using the definition (1.3), Poisson formula takes a form similar to the Riemann-Roch formula, namely:

$$h_{\theta}^0(\overline{E}) - h_{\theta}^0(\overline{E}^{\vee}) = \widehat{\deg} \overline{E}.$$

Here \overline{E}^{\vee} denotes the euclidean lattice dual of \overline{E} and $\widehat{\deg} \overline{E}$ the Arakelov degree of \overline{E} , defined in terms of its of covolume $\text{covol}(\overline{E})$ as

$$\widehat{\deg} \overline{E} = -\log \text{covol}(\overline{E}).$$

The analogy between the invariants (1.2) and (1.3) respectively attached to a vector bundle over a projective curve and to an euclidean lattice, and between the Riemann-Roch and Poisson formulas, belongs to the “well-known facts” of algebraic number theory and is alluded to in classical references such as Tate's thesis ([Tat67], Section 4.2). More recently, it has been reexamined and developed, in relation with Arakelov geometry, in the works of Roessler ([Roe93]), van der Geer and Schoof ([vdGS00]) and Groenewegen ([Gro01]).

At this point, it may be worth to emphasize that many contributions to Arakelov geometry consider another invariant of an euclidean lattice $\overline{E} := (E, \|\cdot\|)$ as the counterpart of the dimension (1.2) of the space of sections of vector bundles over curves — namely the real number

$$h_{\text{Ar}}^0(\overline{E}) := \log |\{v \in E \mid \|v\| \leq 1\}|.$$

This definition already appears, in substance, in Weil's famous note [Wei39] (which, somewhat surprisingly, does not allude to the analogy between (1.2) and (1.3)), and explicitly in the first expositions of Arakelov geometry, notably in [Szp85] and [Man85] (see also [GMS91]).

1.1.3. In this article, to a pro-euclidean lattice

$$\widehat{\overline{E}} := (\widehat{E}, E_{\mathbb{R}}^{\text{Hilb}}, \|\cdot\|),$$

we attach some infinite dimensional generalizations of the invariant $h_{\theta}^0(\overline{E})$ previously defined for (finite dimensional) euclidean lattices.

Notably we consider the invariant in $[0, +\infty]$:

$$(1.4) \quad \underline{h}_{\theta}^0(\widehat{\overline{E}}) := \log \sum_{v \in \widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}} e^{-\pi \|v\|^2}$$

and we investigate some classes of pro-euclidean euclidean lattices for which this invariant is finite and may be computed as the limit

$$\lim_{i \rightarrow +\infty} h_{\theta}^0(\overline{E}_i)$$

of the θ -invariants of the euclidean lattices in a projective system \overline{E}_{\bullet} that defines $\widehat{\overline{E}}$ as in 1.1.1.

²Hilbert's formulation of his “twelfth problem”, together with Hecke's work on quadratic reciprocity over general number fields, might have been an earlier hint toward this correspondence; see [Hil00] p. 278-279, and [Hec19], [Hec23] Kapitel VIII.

As a preliminary to our study of infinite dimensional euclidean lattices, we also establish various properties of the invariant $h_\theta^0(\overline{E})$ in the finite dimensional case. A key point is the subadditivity of h_θ^0 , already noticed in [Gro01]. Namely, *for any admissible³ short exact sequence of euclidean lattices*

$$0 \longrightarrow \overline{E} \xrightarrow{i} \overline{F} \xrightarrow{p} \overline{G} \longrightarrow 0,$$

the following inequality holds:

$$(1.5) \quad h_\theta^0(\overline{F}) \leq h_\theta^0(\overline{E}) + h_\theta^0(\overline{G}).$$

A similar subadditivity is clearly satisfied by the integer valued invariant $h^0(C, \cdot)$ defined by (1.2) in the geometric case. It is remarkable that it holds *ne varietur* in the arithmetic case — in the form (1.5), that involves no error term depending on the ranks of \overline{E} , \overline{F} and \overline{G} . This “unexpectedly good” behavior of h_θ^0 is crucial to extend it to the infinite dimensional situation. Actually, in the last sections of this article, we introduce a suitable notion of admissible short exact sequence of pro-euclidean lattices and we show that, for any such sequence

$$0 \longrightarrow \widehat{\overline{E}} \xrightarrow{i} \widehat{\overline{F}} \xrightarrow{p} \widehat{\overline{G}} \longrightarrow 0,$$

the following extension of (1.5) still holds:

$$\underline{h}_\theta^0(\widehat{\overline{F}}) \leq \underline{h}_\theta^0(\widehat{\overline{E}}) + \underline{h}_\theta^0(\widehat{\overline{G}}).$$

In a forthcoming part of this work, we shall show that such estimates between θ -invariants attached to short exact sequences of pro-euclidean lattices have interesting applications to transcendence theory and Diophantine geometry.

1.2. Contents. We now proceed to a synopsis of the main results of this article. More detailed presentations of these are given in the first paragraphs of each section.

1.2.1. Section 2 collects basic facts concerning hermitian vector bundles over an arithmetic curve $\text{Spec } \mathcal{O}_K$ (when $K = \mathbb{Q}$ and $\mathcal{O}_K = \mathbb{Z}$, they are simply the euclidean lattices previously considered in this introduction) and could be skipped by readers familiar with Arakelov geometry.

The θ -invariants of hermitian vector bundles over arithmetic curves are studied in sections 3 and 4.

In section 3, we define the invariants $h_\theta^0(\overline{E})$ and $h_\theta^1(\overline{E})$ attached to some hermitian vector bundle over a general arithmetic curve $\text{Spec } \mathcal{O}_K$ and we present some of their basic properties, notably the Poisson-Riemann-Roch formula and their monotonicity properties. We also establish upper bounds for the θ -invariants of hermitian line bundles over $\text{Spec } \mathcal{O}_K$, and we discuss the subadditivity property (1.5) and its variants. This section pursue the previous work on θ -invariants in [vdGS00] and [Gro01], but our presentation is self-contained and does not require any familiarity with these articles.

In section 4, we focus on the situation where $\mathcal{O}_K = \mathbb{Z}$, and we establish diverse properties of the θ -invariants of some euclidean lattice \overline{E} that compare them to diverse invariants of \overline{E} classically considered in geometry of numbers. Notably we establish the following estimates comparing the two “arithmetic avatars” $h_\theta^0(\overline{E})$ and $h_{\text{Ar}}^0(\overline{E})$ of the “geometric” invariant $h^0(C, V)$:

$$(1.6) \quad -\pi \leq h_\theta^0(\overline{E}) - h_{\text{Ar}}^0(\overline{E}) \leq (\text{rk } E/2) \cdot \log \text{rk } E - \log(1 - 1/2\pi),$$

where $\text{rk } E$ denotes the rank of \overline{E} .

Our proof of (1.6) uses some estimates established by Banaszczyk in his work [Ban93] devoted to the derivation of “transference inequalities”, relating invariants of euclidean lattices and of the dual

³By definition, this means that $0 \longrightarrow E \xrightarrow{i} F \xrightarrow{p} G \longrightarrow 0$ is a short exact sequence of \mathbb{Z} -modules and that $i_{\mathbb{R}}$ and the transpose of $p_{\mathbb{R}}$ are isometries with respect to the euclidean norms on $E_{\mathbb{R}}$, $F_{\mathbb{R}}$, and $G_{\mathbb{R}}$ defining the euclidean lattices \overline{E} , \overline{F} and \overline{G} .

lattices, with essentially optimal constants. Banaszczyk's techniques rely on the properties of the measure

$$\gamma_{\overline{E}} := \sum_{v \in E} e^{-\pi \|v\|^2} \delta_v$$

supported by the lattice E inside $E_{\mathbb{R}}$ and of its Fourier transform, which is a C^∞ and E^\vee -periodic function on $E_{\mathbb{R}}^\vee$. Although these techniques have been developed without reference to the analogy between number fields and function fields or to the invariants $h_\theta^0(\overline{E})$ or $h_\theta^1(\overline{E}) := h_\theta^0(\overline{E}^\vee)$, they constitute a remarkably useful tool to investigate the properties of the θ -invariants.

In a related vein to the comparison estimates (1.6), we introduce the following extension of $h_{\text{Ar}}^0(\overline{E})$, defined for every $t \in \mathbb{R}_+^*$:

$$h_{\text{Ar}}^0(\overline{E}, t) := \log |\{v \in E \mid \|v\|^2 \leq t\}|,$$

and we prove that it admits an asymptotic version

$$\tilde{h}_{\text{Ar}}^0(\overline{E}, t) := \tilde{h}_{\text{Ar}}^0(\overline{E}, t) := \lim_{n \rightarrow +\infty} \frac{1}{n} h_{\text{Ar}}^0(\overline{E}^{\oplus n}, nt)$$

— the limit exists in \mathbb{R}_+ — that, remarkably enough, basically coincides with the Legendre transform of the function $\log \theta_{\overline{E}}$, where

$$\theta_{\overline{E}}(t) := \sum_{v \in E} e^{-\pi t \|v\|^2}.$$

Namely, we establish the dual relations:

$$(1.7) \quad \text{for every } x \in \mathbb{R}_+^*, \tilde{h}_{\text{Ar}}^0(\overline{E}, x) = \inf_{\beta > 0} (\pi \beta x + \log \theta_{\overline{E}}(\beta))$$

and

$$(1.8) \quad \text{for every } \beta \in \mathbb{R}_+^*, \log \theta_{\overline{E}}(\beta) = \sup_{x > 0} (\tilde{h}_{\text{Ar}}^0(\overline{E}, x) - \pi \beta x).$$

These relations appear as a special case of a version of Cramér's theory of large deviations valid over some measure space of infinite total mass. This theory is presented in Appendix A and turns out to be closely related to the thermodynamic formalism.

1.2.2. The remaining sections are devoted to the construction of categories of hermitian vector bundles of (possibly) infinite rank and to the definition and the investigation of their θ -invariants.

Section 5 introduces some categories CT_A and CTC_A of (topological) A -modules over a Dedekind ring A . When A is the ring of integers \mathcal{O}_K of some number field K , objects of the categories CT_A and CTC_A will form the “algebraic constituents” of the infinite dimensional hermitian vector bundles over $\text{Spec } \mathcal{O}_K$ that are the subject of this article.

Diverse proofs of section 5 rely on some “automatic continuity” results, discussed in Appendix B, concerning R -linear maps $\phi : R^{\mathbb{N}} \rightarrow R$, for suitable rings R , when $R^{\mathbb{N}} \simeq \varprojlim_n R^n$ is equipped with the pro-discrete topology.

The infinite dimensional hermitian vector bundles and their morphisms are formally introduced and studied in section 6. In section 7, we define and establish the basic properties of their θ -invariants.

In section 8, we establish the main technical results of this article concerning θ -invariants of pro-hermitian vector bundles. A special case of our results may be summarized as follows.

If $\overline{E} := (E, \|\cdot\|)$ denotes some euclidean lattice, then, for every real number λ , we may consider the euclidean lattice

$$\overline{E} \otimes \overline{\mathcal{O}}(\lambda) := (E, e^{-\lambda} \|\cdot\|)$$

deduced from \overline{E} by scaling its metric by a factor $e^{-\lambda}$.

Let us consider a projective system \overline{E}_\bullet of euclidean lattices and the attached pro-euclidean lattice $\varprojlim_i \overline{E}_i$, as in 1.1.1 above. This pro-euclidean lattice may be thought as being constructed from \overline{E}_0 as “successive extensions” by the euclidean lattices

$$\overline{S}_i := (\ker q_i, \|\cdot\|_{i|\ker q_i, \mathbb{R}})$$

which are defined as the kernels of the morphisms q_i , $i \in \mathbb{N}$. We shall show that, *if, for some $\epsilon \in \mathbb{R}_+^*$, the summability condition*

$$\sum_{i \in \mathbb{N}} h_\theta^0(\overline{S}_i \otimes \overline{\mathcal{O}}(\epsilon)) < +\infty$$

is satisfied, the invariant $h_\theta^0(\varprojlim_i \overline{E}_i)$ (defined by (1.4)) is finite and is given as the limit:

$$(1.9) \quad h_\theta^0(\varprojlim_i \overline{E}_i) = \lim_{i \rightarrow \infty} h_\theta^0(\overline{E}_i).$$

The proofs of this finiteness and of the equality (1.9) involve measure theoretic arguments on the Polish space defined by $\varprojlim_i E_i$ equipped with the pro-discrete topology, concerning the convergence properties of the sequence of Banaszczyk’s measures

$$\gamma_{\overline{E}_i} := \sum_{v \in E_i} e^{-\pi \|v\|_i^2} \delta_v.$$

Various facts concerning measure theory on Polish spaces constructed as projective limits of countable systems of countable discrete sets are presented in Appendix C in a form suited to the derivation of the results in section 8.

Our results allow us to define natural classes of pro-hermitian vector bundles over an arithmetic curve $\text{Spec } \mathcal{O}_K$ with finite and well-behaved θ -invariants: the *strongly summable* and the *θ -finite* pro-hermitian vector bundles.

Finally, in section 9, we investigate short exact sequences of pro-hermitian vector bundles and we extend the subadditivity properties of h_θ^0 to this infinite dimensional setting, and give some applications to strongly summable and θ -finite pro-hermitian vector bundles.

1.3. Acknowledgements. It is a pleasure to thank François Charles for numerous discussions about the results in this article and their applications, and for his comments on a preliminary version.

I am much indebted to Gaëtan Chenevier for making me aware of the “automatic continuity” results discussed in Appendix B and I thank Antoine Chambert-Loir for the reference [Spe50]. This allowed me to present the results in Section 5.4 in their natural generality⁴.

At an early stage of the writing of this article, I had the opportunity to present some applications of the formalism it develops to Antoine Chambert-Loir, François Charles and Gerard Freixas, and I would like to thank them for their encouraging comments.

Several years ago, Alain Connes pointed out to me that the invariant $h_\theta^0(\overline{E})$ of an euclidean lattice $\overline{E} = (E, \|\cdot\|)$ should be expressed in terms of the measure $(p(e))_{e \in E}$ on E of a given energy $\sum_{e \in E} p(e) \|e\|^2$ for which the information theoretic entropy $-\sum_{e \in E} p(e) \log p(e)$ is maximal. This suggestion, now formalized in Proposition 4.4.7 *infra*, has been crucial in arousing my interest in the relations between θ -invariants of euclidean lattices and concepts from statistical thermodynamics, and I warmly thank Alain Connes for sharing his insight.

⁴An earlier version, relying on the arguments discussed after the proof of Proposition 5.4.10, had to make a countability assumption on A instead of condition **Ded₃**.

1.4. Notation and conventions. We denote by \mathbb{N} the set of non-negative integers and, for any $k \in \mathbb{N}$, we denote by $\mathbb{N}_{>k}$ (resp., by $\mathbb{N}_{\geq k}$) the set of non-negative integers larger than k (resp., larger than or equal to k).

By *countable*, we mean “of cardinality at most the cardinality of \mathbb{N} .”

If M is a module over some ring A , and if B is some commutative A -algebra, we denote by M_B the “base changed” module $M \otimes_A B$. Similarly, if $\phi : M \rightarrow N$ is a morphism of A -modules, we let:

$$\phi_B := \phi \otimes_A Id_B : M_B \rightarrow N_B.$$

2. HERMITIAN VECTOR BUNDLES OVER ARITHMETIC CURVES

In this section, we gather some basic results concerning hermitian vector bundles over arithmetic curves (that is, the affine schemes defined by the rings of integers of number fields). These results are well known, with the exception of the content of Section 2.4.7, and appear for instance in [Szp85], [Neu92], [BGS94], [Sou97], [Bos96], and [BK10].

This section is primarily intended as a reference section. It might be skipped by readers familiar with Arakelov geometry, who could refer to it only when needed.

We denote by K a number field, by \mathcal{O}_K its ring of integers, and by

$$\pi : \text{Spec } \mathcal{O}_K \rightarrow \text{Spec } \mathbb{Z}$$

the morphism of schemes from $\text{Spec } \mathcal{O}_K$ to $\text{Spec } \mathbb{Z}$, defined by the inclusion morphism $\mathbb{Z} \hookrightarrow \mathcal{O}_K$.

2.1. Definitions and basic operations.

2.1.1. Hermitian vector bundles over arithmetic curves. A *hermitian vector bundle* over $\text{Spec } \mathcal{O}_K$ is a pair

$$\overline{E} = (E, (\|\cdot\|_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}})$$

consisting in a finitely generated projective \mathcal{O}_K -module E and in a family $(\|\cdot\|_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}}$ of hermitian norms $\|\cdot\|_\sigma$ on the complex vector spaces

$$E_\sigma := E \otimes_{\mathcal{O}_K, \sigma} \mathbb{C}$$

defined by means of the field embeddings $\sigma : K \hookrightarrow \mathbb{C}$. The family $(\|\cdot\|_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}}$ is moreover required to be invariant under complex conjugation.⁵

When $K = \mathbb{Q}$, a hermitian vector bundle $\overline{E} = (E, \|\cdot\|)$ over $\text{Spec } \mathcal{O}_K = \text{Spec } \mathbb{Z}$ may be identified with a *euclidean lattice*, defined by a free \mathbb{Z} -module of finite rank E and a euclidean norm $\|\cdot\|$ on $E_\mathbb{R} := E \otimes_\mathbb{Z} \mathbb{R}$. (Indeed, for any such E , the data of some hermitian norm on $E_\mathbb{C} := E \otimes_\mathbb{Z} \mathbb{C}$ invariant under complex conjugation and of some euclidean norm on $E_\mathbb{R}$ are equivalent.)

The *rank* of some hermitian vector bundle \overline{E} as above is the rank of the \mathcal{O}_K -module E , or equivalently the dimension of the complex vector spaces E_σ . A *hermitian line bundle* is a hermitian vector bundle of rank one.

When confusion may arise, the family of hermitian norms underlying some hermitian vector bundle \overline{E} over $\text{Spec } \mathcal{O}_K$ will be denoted by $(\|\cdot\|_{\overline{E}, \sigma})_{\sigma:K \hookrightarrow \mathbb{C}}$.

An *isometric isomorphism*, or simply an *isomorphism*, between two hermitian vector bundles \overline{E} and \overline{F} over $\text{Spec } \mathcal{O}_K$ is an isomorphism $\psi : E \xrightarrow{\sim} F$ between the underlying \mathcal{O}_K -modules which, after every base change $\sigma : K \hookrightarrow \mathbb{C}$, defines an isometry of complex normed vector spaces between $(E_\sigma, \|\cdot\|_{\overline{E}, \sigma})$ and $(F_\sigma, \|\cdot\|_{\overline{F}, \sigma})$.

⁵Namely, for every embedding $\sigma : K \hookrightarrow \mathbb{C}$, we may consider the complex conjugate embedding $\overline{\sigma} : K \hookrightarrow \mathbb{C}$ and the \mathbb{C} -antilinear isomorphism $F_\infty : E_\sigma = E \otimes_{\mathcal{O}_K, \sigma} \mathbb{C} \xrightarrow{\sim} E_{\overline{\sigma}} = E \otimes_{\mathcal{O}_K, \overline{\sigma}} \mathbb{C}$ defined by $F_\infty(e \otimes \lambda) = e \otimes \overline{\lambda}$. The hermitian norms $\|\cdot\|_\sigma$ and $\|\cdot\|_{\overline{\sigma}}$ have to satisfy: $\|F_\infty(\cdot)\|_{\overline{\sigma}} = \|\cdot\|_\sigma$.

2.1.2. *Pull back. Tensor operations.* Let L be a number field extension of K . The inclusion $\mathcal{O}_K \hookrightarrow \mathcal{O}_L$ defines a morphism of arithmetic curves

$$f : \operatorname{Spec} \mathcal{O}_L \longrightarrow \operatorname{Spec} \mathcal{O}_K.$$

The *pull-back* $f^* \overline{E}$ is defined as the hermitian vector bundle over $\operatorname{Spec} \mathcal{O}_L$

$$f^* \overline{E} := (f^* E, (\|\cdot\|_\tau)_{\tau: L \hookrightarrow \mathbb{C}})$$

where $f^* E := E \otimes_{\mathcal{O}_K} \mathcal{O}_L$ and where, for every field embedding $\tau : L \hookrightarrow \mathbb{C}$, of restriction $\tau|_K =: \sigma$, $\|\cdot\|_\tau$ denotes the hermitian norm $\|\cdot\|_\sigma$ on

$$(f^* E)_\tau := (E \otimes_{\mathcal{O}_K} \mathcal{O}_L) \otimes_{\mathcal{O}_K, \sigma} \mathbb{C} \simeq E \otimes_{\mathcal{O}_K, \sigma} \mathbb{C} =: E_\sigma.$$

To any hermitian vector bundle \overline{E} over $\operatorname{Spec} \mathcal{O}_K$, we may associate its dual hermitian vector bundle \overline{E}^\vee and its exterior powers $\wedge^k \overline{E}$, $k \in \mathbb{N}$. They are defined by means of the (compatible) constructions of duals and exterior powers for projective \mathcal{O}_K -modules and hermitian complex vector spaces.

Similarly, for any two hermitian vector bundles \overline{E} and \overline{F} over $\operatorname{Spec} \mathcal{O}_K$, we may construct their direct sum $\overline{E} \oplus \overline{F}$ and their tensor product $\overline{E} \otimes \overline{F}$ as hermitian vector bundles over $\operatorname{Spec} \mathcal{O}_K$.

These tensor operations are compatible with the pull-back construction defined above. Namely, we have canonical identifications $(f^* \overline{E})^\vee \simeq f^*(\overline{E}^\vee)$, $f^*(\wedge^k \overline{E}) \simeq \wedge^k (f^* \overline{E})$,...

2.1.3. *The hermitian line bundles $\overline{\mathcal{O}}(\delta)$.* For any $\delta \in \mathbb{R}$, we may consider the hermitian line bundle over $\operatorname{Spec} \mathbb{Z}$ defined by

$$\overline{\mathcal{O}}(\delta) := (\mathbb{Z}, \|\cdot\|_{\overline{\mathcal{O}}(\delta)}), \text{ where } \|1\|_{\overline{\mathcal{O}}(\delta)} := e^{-\delta}.$$

We may also consider its pull-back by the morphism $\pi : \operatorname{Spec} \mathcal{O}_K \longrightarrow \operatorname{Spec} \mathbb{Z}$:

$$\overline{\mathcal{O}}_{\operatorname{Spec} \mathcal{O}_K}(\delta) := \pi^* \overline{\mathcal{O}}(\delta).$$

For any hermitian vector bundle \overline{E} over $\operatorname{Spec} \mathcal{O}_K$, the hermitian vector bundle $\overline{E} \otimes \overline{\mathcal{O}}_{\operatorname{Spec} \mathcal{O}_K}(\delta)$ may be identified with the hermitian vector bundle which admits the same underlying \mathcal{O}_K -module E as \overline{E} , and whose hermitian structure is defined by the hermitian norms defining \overline{E} scaled by the factor $e^{-\delta}$.

For simplicity, we shall often write $\overline{E} \otimes \overline{\mathcal{O}}(\delta)$ instead of $\overline{E} \otimes \overline{\mathcal{O}}_{\operatorname{Spec} \mathcal{O}_K}(\delta)$.

The “trivial” hermitian line bundle — namely, $\overline{\mathcal{O}}_{\operatorname{Spec} \mathcal{O}_K}(0)$ — will also be denoted by $\overline{\mathcal{O}}_{\operatorname{Spec} \mathcal{O}_K}$ and, when no confusion may arise, simply by $\overline{\mathcal{O}}$.

2.2. Direct images. The canonical hermitian line bundle $\overline{\omega}_{\mathcal{O}_K/\mathbb{Z}}$ over $\operatorname{Spec} \mathcal{O}_K$. 6.1.5

2.2.1. To any hermitian vector bundle $\overline{E} = (E, (\|\cdot\|_\sigma)_{\sigma: K \hookrightarrow \mathbb{C}})$ is attached its *direct image*⁶ $\pi_* \overline{E}$ over $\operatorname{Spec} \mathbb{Z}$.

Observe that we have an isomorphism of \mathbb{C} -algebras:

$$\begin{aligned} \mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{C} &\xrightarrow{\sim} \bigoplus_{\sigma: K \hookrightarrow \mathbb{C}} \mathbb{C} \\ \alpha \otimes \lambda &\longmapsto (\sigma(\alpha)\lambda)_{\sigma: K \hookrightarrow \mathbb{C}}. \end{aligned}$$

⁶The construction of direct images of hermitian vector bundles introduced here makes actually sense in a considerably more general setting; see [BK10], Section 1.2.1. Notably, it may be extended to any morphism of arithmetic curves $f : \operatorname{Spec} \mathcal{O}_L \longrightarrow \operatorname{Spec} \mathcal{O}_K$.

Therefore, for any \mathcal{O}_K -module M , if $\pi_* M$ denotes the underlying \mathbb{Z} -module, we have

$$\begin{aligned} (\pi_* M)_{\mathbb{C}} &:= \pi_* M \otimes_{\mathbb{Z}} \mathbb{C} \\ &\simeq \pi_* M \otimes_{\mathcal{O}_K} (\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{C}) \\ &\simeq \bigoplus_{\sigma: K \hookrightarrow \mathbb{C}} (M \otimes_{\mathcal{O}_K, \sigma} \mathbb{C}). \end{aligned}$$

Using this observation, the direct image $\pi_* \overline{E}$ may be defined as the hermitian vector bundle of rank $[K : \mathbb{Q}] \cdot \text{rk } E$ over $\text{Spec } \mathbb{Z}$:

$$\pi_* \overline{E} := (\pi_* E, \|\cdot\|_{\pi_* \overline{E}})$$

where, for any $v = (v_{\sigma})_{\sigma: K \hookrightarrow \mathbb{C}}$ in $(\pi_* E)_{\mathbb{C}} \simeq \bigoplus_{\sigma: K \hookrightarrow \mathbb{C}} E_{\sigma}$,

$$\|v\|_{\pi_* \overline{E}}^2 := \sum_{\sigma: K \hookrightarrow \mathbb{C}} \|v_{\sigma}\|_{\sigma}^2.$$

Clearly we have:

$$\text{rk } \pi_* E = [K : \mathbb{Q}] \cdot \text{rk } E.$$

2.2.2. The compatibility of the direct image operation π_* and of the duality of hermitian vector bundles involves the *canonical hermitian line bundle* $\overline{\omega}_{\mathcal{O}_K/\mathbb{Z}} := (\omega_{\mathcal{O}_K/\mathbb{Z}}, (\|\cdot\|_{\sigma})_{\sigma: K \hookrightarrow \mathbb{C}})$ over $\text{Spec } \mathcal{O}_K$. It is defined as the “canonical module”

$$\omega_{\mathcal{O}_K/\mathbb{Z}} := \text{Hom}_{\mathbb{Z}}(\mathcal{O}_K, \mathbb{Z})$$

— also known as the “inverse of the different” — equipped with the hermitian norms defined by

$$\|\text{tr}_{K/\mathbb{Q}}\|_{\sigma} = 1$$

for any embedding $\sigma : K \hookrightarrow \mathbb{C}$, where $\text{tr}_{K/\mathbb{Q}}$ denotes the trace map from K to \mathbb{Q} (it is indeed a non-zero element in $\text{Hom}_{\mathbb{Z}}(\mathcal{O}_K, \mathbb{Z})$).

Indeed, for any hermitian vector bundle \overline{E} over $\text{Spec } \mathcal{O}_K$, we have a canonical (!) isometric isomorphism of hermitian vector bundles over $\text{Spec } \mathbb{Z}$,

$$(2.1) \quad \pi_*(\overline{E}^{\vee} \otimes \overline{\omega}_{\mathcal{O}_K/\mathbb{Z}}) \xrightarrow{\sim} (\pi_* \overline{E})^{\vee}.$$

(See for instance [BK10], Proposition 3.2.2. For any ξ in $E^{\vee} := \text{Hom}_{\mathcal{O}_K}(E, \mathcal{O}_K)$ and any λ in $\omega_{\mathcal{O}_K/\mathbb{Z}}$, the “relative duality isomorphism” (2.1) maps $\xi \otimes \lambda$ to $\lambda \circ \xi$.)

2.3. Arakelov degree.

2.3.1. *Definition and basic properties.* The *Arakelov degree* of some hermitian line bundle $\overline{L} := (L, (\|\cdot\|_{\sigma})_{\sigma: K \hookrightarrow \mathbb{C}})$ over $\text{Spec } \mathcal{O}_K$ is defined by the quality, valid for any $s \in L \setminus \{0\}$:

$$(2.2) \quad \widehat{\deg} \overline{L} := \log |L/\mathcal{O}_K s| - \sum_{\sigma: K \hookrightarrow \mathbb{C}} \log \|s\|_{\sigma}$$

$$(2.3) \quad = \sum_{\mathfrak{p}} v_{\mathfrak{p}}(s) \log N\mathfrak{p} - \sum_{\sigma: K \hookrightarrow \mathbb{C}} \log \|s\|_{\sigma}.$$

In the last equality, \mathfrak{p} runs over the closed points of $\text{Spec } \mathcal{O}_K$ — that is, on the non-zero prime ideals of \mathcal{O}_K — and $v_{\mathfrak{p}}(s)$ denotes the \mathfrak{p} -adic valuation of s , seen as a section of the invertible sheaf over $\text{Spec } \mathcal{O}_K$ associated to L . Moreover, $N\mathfrak{p} := |\mathcal{O}_K/\mathfrak{p}|$ denotes the norm of \mathfrak{p} .

This definition is extended to hermitian vector bundles of arbitrary rank over $\text{Spec } \mathcal{O}_K$ by means of the formula:

$$\widehat{\deg} \overline{E} := \widehat{\deg} \wedge^{\text{rk } E} \overline{E}.$$

If \overline{E} is an hermitian vector bundle over $\text{Spec } \mathbb{Z}$, or in other words, a euclidean lattice, its Arakelov degree may be expressed in term of its covolume⁷ $\text{covol}(\overline{E})$; namely:

$$\widehat{\deg} \overline{E} = -\log \text{covol}(\overline{E}).$$

For any two hermitian line bundles \overline{L}_1 and \overline{L}_2 over $\text{Spec } \mathcal{O}_K$, the expression (2.3) for their Arakelov degree shows that

$$(2.4) \quad \widehat{\deg} (\overline{L}_1 \otimes \overline{L}_2) = \widehat{\deg} \overline{L}_1 + \widehat{\deg} \overline{L}_2.$$

For any two hermitian vector bundles \overline{E}_1 and \overline{E}_2 over $\text{Spec } \mathcal{O}_K$, there is a canonical⁸ isomorphism of hermitian line bundles over $\text{Spec } \mathcal{O}_K$:

$$\wedge^{\text{rk}(E_1 \oplus E_2)} (\overline{E}_1 \oplus \overline{E}_2) \xrightarrow{\sim} \wedge^{\text{rk } E_1} \overline{E}_1 \otimes \wedge^{\text{rk } E_2} \overline{E}_2,$$

and the additivity relation (2.4) applied to $\overline{L}_i := \wedge^{\text{rk } E_i} \overline{E}_i$ ($i = 1, 2$) takes the form:

$$(2.5) \quad \widehat{\deg} (\overline{E}_1 \oplus \overline{E}_2) = \widehat{\deg} \overline{E}_1 + \widehat{\deg} \overline{E}_2.$$

Similarly, by reducing to the case of hermitian line bundles, one shows that, for any hermitian vector bundle \overline{E} over $\text{Spec } \mathcal{O}_K$, we have:

$$\widehat{\deg} \overline{E}^\vee = -\widehat{\deg} \overline{E}.$$

2.3.2. Arakelov degree and direct image. The very definition of the discriminant Δ_K of a number field K shows that

$$\text{covol}(\pi_* \overline{\mathcal{O}}_{\text{Spec } \mathcal{O}_K}) = |\Delta_K|^{1/2}.$$

In other words,

$$(2.6) \quad \widehat{\deg} \pi_* \overline{\mathcal{O}}_{\text{Spec } \mathcal{O}_K} = -\frac{1}{2} \log |\Delta_K|.$$

More generally, for any hermitian vector bundle \overline{E} over $\text{Spec } \mathcal{O}_K$, we have:

$$(2.7) \quad \widehat{\deg} \pi_* \overline{E} = \widehat{\deg} \overline{E} - \frac{1}{2} \log |\Delta_K| \cdot \text{rk } E.$$

Observe that the compatibility of the “relative duality isomorphism” (2.1) and of the “relative Riemann-Roch formula” (2.7) applied to $\overline{E} = \overline{\mathcal{O}}_{\text{Spec } \mathcal{O}_K}$ and $\overline{E} = \overline{\omega}_{\mathcal{O}_K/\mathbb{Z}}$ shows that

$$\widehat{\deg} \overline{\omega}_{\mathcal{O}_K/\mathbb{Z}} = \log |\Delta_K|.$$

2.4. Morphisms and extensions of hermitian vector bundles.

2.4.1. The filtration $\text{Hom}_{\overline{\mathcal{O}}_K}^{\leq \bullet}(\overline{E}, \overline{F})$ on $\text{Hom}_{\mathcal{O}_K}(E, F)$. For any two hermitian vector bundles \overline{E} and \overline{F} on $\text{Spec } \mathcal{O}_K$ and for any $\lambda \in \mathbb{R}_+$, we may introduce the subset $\text{Hom}_{\overline{\mathcal{O}}_K}^{\leq \lambda}(\overline{E}, \overline{F})$ of the \mathcal{O}_K -module $\text{Hom}_{\mathcal{O}_K}(E, F)$ consisting in the \mathcal{O}_K -linear maps

$$\psi : E \longrightarrow F$$

such that, for every $\sigma : K \hookrightarrow \mathbb{C}$, the induced \mathbb{C} -linear map

$$\psi_\sigma : E_\sigma \longrightarrow F_\sigma$$

has an operator norm $\leq \lambda$ when E_σ and F_σ are equipped with the hermitian norms $\|\cdot\|_{\overline{E}_\sigma}$ and $\|\cdot\|_{\overline{F}_\sigma}$.

Besides, for any $\lambda \in \mathbb{R}_+$, transposition defines a bijection:

$$(2.8) \quad \begin{array}{ccc} \text{Hom}_{\overline{\mathcal{O}}_K}^{\leq \lambda}(\overline{E}, \overline{F}) & \xrightarrow{\sim} & \text{Hom}_{\overline{\mathcal{O}}_K}^{\leq \lambda}(\overline{F}^\vee, \overline{E}^\vee) \\ \psi & \longmapsto & \psi^t. \end{array}$$

⁷See for instance Section 3.1, *infra*.

⁸up-to a sign.

Observe that, if \overline{E}_1 , \overline{E}_2 , and \overline{E}_3 are hermitian vector bundles over $\text{Spec } \mathcal{O}_K$, for any $(\lambda_1, \lambda_2) \in \mathbb{R}_+^2$, the composition of an element ψ_1 in $\text{Hom}_{\mathcal{O}_K}^{\leq \lambda_1}(\overline{E}_2, \overline{E}_1)$ and of an element ψ_2 in $\text{Hom}_{\mathcal{O}_K}^{\leq \lambda_2}(\overline{E}_3, \overline{E}_2)$ defines an element $\psi_1 \circ \psi_2$ in $\text{Hom}_{\mathcal{O}_K}^{\leq \lambda_1 + \lambda_2}(\overline{E}_3, \overline{E}_1)$.

In particular, one endows the class of hermitian vector bundles over $\text{Spec } \mathcal{O}_K$ with a structure of category by defining the set of morphisms from \overline{E} to \overline{F} as $\text{Hom}_{\mathcal{O}_K}^{\leq 1}(\overline{E}, \overline{F})$. The isomorphisms in this category coincide with the ones already introduced in 2.1.1.

2.4.2. Injective and surjective admissible morphisms. Observe that $\text{Hom}_{\mathcal{O}_K}^{\leq 1}(\overline{E}, \overline{F})$ contains the *injective admissible morphisms*, defined as the \mathcal{O}_K -linear maps $\psi : E \rightarrow F$ such that ψ is injective with torsion free cokernel and such that, for any $\sigma : K \hookrightarrow \mathbb{C}$, the \mathbb{C} -linear map $\psi_\sigma : E_\sigma \rightarrow F_\sigma$ is an isometry with respect to the hermitian norms $\|\cdot\|_{\overline{E}, \sigma}$ and $\|\cdot\|_{\overline{F}, \sigma}$.

The set $\text{Hom}_{\mathcal{O}_K}^{\leq 1}(\overline{E}, \overline{F})$ also contains the *surjective admissible morphisms*, defined as the surjective \mathcal{O}_K -linear maps $\psi : E \rightarrow F$ such that, for any field embedding $\sigma : K \hookrightarrow \mathbb{C}$, the norm $\|\cdot\|_{\overline{F}, \sigma}$ on F_σ is the quotient norm deduced from the norm $\|\cdot\|_{\overline{E}, \sigma}$ on E_σ by means of the surjective \mathbb{C} -linear map $\psi_\sigma : E_\sigma \rightarrow F_\sigma$.

The transposition map (2.8), with $\lambda = 1$, exchanges injective and surjective admissible morphisms.

2.4.3. Heights of morphisms. Recall that, to any non-zero K -linear map $\phi : E_K \rightarrow F_K$ is attached its *height with respect to \overline{E} and \overline{F}* , defined as

$$(2.9) \quad \text{ht}(\overline{E}, \overline{F}, \phi) := \sum_{\mathfrak{p}} \log \|\phi\|_{\mathfrak{p}} + \sum_{\sigma: K \hookrightarrow \mathbb{C}} \log \|\phi\|_{\sigma}.$$

(See [Bos01], Section 4.1.4, where $\text{ht}(\overline{E}, \overline{F}, \phi)$ is noted $h(\overline{E}, \overline{F}, \phi)$.) In the right-hand side of (2.9), \mathfrak{p} varies over the maximal ideals of \mathcal{O}_K , and $\|\phi\|_{\mathfrak{p}}$ denotes the \mathfrak{p} -adic norm of $\phi \in \text{Hom}_K(E_K, F_K) \simeq (E^\vee \otimes_{\mathcal{O}_K} F)_K$, defined by the equivalence⁹:

$$\|\phi\|_{\mathfrak{p}} \leq 1 \iff \phi \in (E^\vee \otimes_{\mathcal{O}_K} F)_{\mathcal{O}_{K, \mathfrak{p}}}$$

and the normalization condition:

$$\|\varpi \phi\|_{\mathfrak{p}} = (N\mathfrak{p})^{-1} \|\phi\|_{\mathfrak{p}},$$

where ϖ denotes a uniformizing element of $\mathcal{O}_{K, \mathfrak{p}}$ and $N\mathfrak{p} := |\mathcal{O}_K/\mathfrak{p}|$ the norm of \mathfrak{p} . Besides, $\|\phi\|_{\sigma}$ denotes the operator norm of $\phi_\sigma : E_\sigma \rightarrow F_\sigma$ defined from the hermitian norms $\|\cdot\|_{\overline{E}, \sigma}$ and $\|\cdot\|_{\overline{F}, \sigma}$.

Clearly, for any ϕ in $\text{Hom}_{\mathcal{O}_K}^{\leq 1}(\overline{E}, \overline{F}) \setminus \{0\}$, all the norms $\|\phi\|_{\mathfrak{p}}$ and $\|\phi\|_{\sigma}$ belong to $]0, 1]$, and therefore $h(\overline{E}, \overline{F}, \phi)$ belongs to \mathbb{R}_- .

Besides, when \overline{E} and \overline{F} are hermitian line bundles, then for any non-zero element ϕ of the K -vector space $\text{Hom}_K(E_K, F_K)$, the very definition (2.9) of its height shows that

$$\widehat{\deg}(\overline{E}^\vee \otimes \overline{F}) = -\text{ht}(\overline{E}, \overline{F}, \phi),$$

or equivalently:

$$(2.10) \quad \text{ht}(\overline{E}, \overline{F}, \phi) = \widehat{\deg} \overline{E} - \widehat{\deg} \overline{F}.$$

2.4.4. Admissible short exact sequences of hermitian vector bundles. An *admissible short exact sequence* (also called an *admissible extension*) of hermitian vector bundles over $\text{Spec } \mathcal{O}_K$ is a diagram

$$(2.11) \quad \overline{\mathcal{E}} : 0 \rightarrow \overline{E} \xrightarrow{i} \overline{F} \xrightarrow{p} \overline{G} \rightarrow 0,$$

where $\overline{E} = (E, (\|\cdot\|_{\overline{E}, \sigma})_{\sigma: K \hookrightarrow \mathbb{C}})$, $\overline{F} = (F, (\|\cdot\|_{\overline{F}, \sigma})_{\sigma: K \hookrightarrow \mathbb{C}})$, and $\overline{G} = (G, (\|\cdot\|_{\overline{G}, \sigma})_{\sigma: K \hookrightarrow \mathbb{C}})$ are hermitian vector bundles over $\text{Spec } \mathcal{O}_K$, and where

$$\mathcal{E} : 0 \rightarrow E \xrightarrow{i} F \xrightarrow{p} G \rightarrow 0$$

⁹We denote by $\mathcal{O}_{K, \mathfrak{p}}$ the discrete valuation ring defined as the localization of \mathcal{O}_K at \mathfrak{p} .

is a short exact sequence of \mathcal{O}_K -modules such that, for any field embedding $\sigma : K \hookrightarrow \mathbb{C}$, the short exact sequence of complex vector spaces

$$0 \longrightarrow E_\sigma \xrightarrow{i_\sigma} F_\sigma \xrightarrow{p_\sigma} G_\sigma \longrightarrow 0$$

(deduced from \mathcal{E} by the base change σ) is compatible with the hermitian norms $\|\cdot\|_{\overline{E},\sigma}$, $\|\cdot\|_{\overline{F},\sigma}$ and $\|\cdot\|_{\overline{G},\sigma}$.¹⁰

The additivity of the Arakelov degree for direct sums (2.5) extends to admissible short exact sequences. Namely, for any admissible short exact sequence (2.11) of hermitian vector bundles over $\text{Spec } \mathcal{O}_K$, the following equality holds:

$$(2.12) \quad \widehat{\deg} \overline{F} = \widehat{\deg} \overline{E} + \widehat{\deg} \overline{G}.$$

This follows from the existence of a canonical¹¹ isomorphism of hermitian line bundles

$$\wedge^{\text{rk } F} \overline{F} \simeq \wedge^{\text{rk } E} \overline{E} \otimes \wedge^{\text{rk } G} \overline{G}$$

attached to the short exact sequence (2.11).

An admissible short exact sequence of hermitian vector bundles (2.11) is said to be *split* when there exists an admissible injective morphism of hermitian vector bundles

$$s : \overline{G} \longrightarrow \overline{F}$$

such that $p \circ s = \text{Id}_G$. The morphism s is then uniquely determined.

Indeed, for every embedding $\sigma : K \hookrightarrow \mathbb{C}$, the \mathbb{C} -linear map $s_\sigma : G_\sigma \longrightarrow F_\sigma$ deduced from s by the extension of scalars $\sigma : \mathcal{O}_K \longrightarrow \mathbb{C}$ necessarily coincides with the *orthogonal splitting* s_σ^\perp of the short exact sequence of complex vector spaces

$$(2.13) \quad \mathcal{E}_\sigma : 0 \longrightarrow E_\sigma \xrightarrow{i_\sigma} F_\sigma \xrightarrow{p_\sigma} G_\sigma \longrightarrow 0.$$

attached to the hermitian metric $\|\cdot\|_{\overline{F},\sigma}$ on F_σ . (Recall that s_σ^\perp is defined as the (linear) section s_σ^\perp of p_σ which maps G_σ isomorphically onto the orthogonal complement $i_\sigma(E_\sigma)^\perp$ of $i_\sigma(E_\sigma)$ in F_σ equipped with the hermitian metric $\|\cdot\|_{\overline{F},\sigma}$.)

2.4.5. Arithmetic extensions. Let E and G be two finitely generated projective \mathcal{O}_K -modules. In [BK10] is introduced the *arithmetic extension group* $\widehat{\text{Ext}}_{\mathcal{O}_K}^1(G, E)$ of G by E . In the setting of the present paper, where we work over arithmetic curves of the form $\text{Spec } \mathcal{O}_K$, it may be defined as:

$$(2.14) \quad \widehat{\text{Ext}}_{\mathcal{O}_K}^1(G, E) := \text{Hom}_{\mathcal{O}_K}(G, E) \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \xrightarrow{\sim} \frac{(\bigoplus_{\sigma: K \hookrightarrow \mathbb{C}} \text{Hom}_{\mathbb{C}}(G_\sigma, E_\sigma))^{F_\infty}}{\text{Hom}_{\mathcal{O}_K}(G, E)}.$$

(See [BK10], Example 2.2.3. As usual, F_∞ denotes the complex conjugation.)

The group $\widehat{\text{Ext}}_{\mathcal{O}_K}^1(G, E)$ classifies, up to isomorphism, the *arithmetic extensions* of G by E . Recall that such an arithmetic extension is a pair $(\mathcal{E}, (s(\sigma))_{\sigma: K \hookrightarrow \mathbb{C}})$ where

$$\mathcal{E} : 0 \longrightarrow E \xrightarrow{i} F \xrightarrow{p} G \longrightarrow 0$$

is an extension (of \mathcal{O}_K -modules) of G by E , and where $(s(\sigma))_{\sigma: K \hookrightarrow \mathbb{C}}$ is a family, invariant under complex conjugation, of \mathbb{C} -linear splittings $s(\sigma) : G_\sigma \longrightarrow F_\sigma$ of splittings of the extension of complex vector spaces

$$\mathcal{E}_\sigma : 0 \longrightarrow E_\sigma \xrightarrow{i_\sigma} F_\sigma \xrightarrow{p_\sigma} G_\sigma \longrightarrow 0.$$

¹⁰Namely, i_σ is required to be an isometry for the norms $\|\cdot\|_{\overline{E},\sigma}$ and $\|\cdot\|_{\overline{F},\sigma}$, and the norm $\|\cdot\|_{\overline{G},\sigma}$ is required to be the quotient norm deduced from $\|\cdot\|_{\overline{F},\sigma}$ by means of p_σ . Equivalently, i and p are respectively injective and surjective admissible morphisms from \overline{E} to \overline{F} and from \overline{F} to \overline{G} .

¹¹up-to a sign.

Indeed, observe that, for any arithmetic extension as above, one may choose some \mathcal{O}_K -linear splitting $s^{\text{int}} \in \text{Hom}_{\mathcal{O}_K}(G, F)$ of \mathcal{E} . Then, for any $\sigma : K \hookrightarrow \mathbb{C}$,

$$p_\sigma \circ (s(\sigma) - s_\sigma^{\text{int}}) = 0,$$

and therefore there exists a uniquely determined $T_\sigma \in \text{Hom}_{\mathbb{C}}(G_\sigma, E_\sigma)$ such that

$$s(\sigma) - s_\sigma^{\text{int}} = i_\sigma \circ T_\sigma.$$

Let us choose an integral splitting of (2.11), that is an \mathcal{O}_K -linear section $s^{\text{int}} : G \rightarrow F$ of $p : F \rightarrow G$.

$$T_\sigma = s_\sigma^\perp - s_\sigma^{\text{int}}.$$

The family $(T_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}}$ belongs to $(\bigoplus_{\sigma:K \hookrightarrow \mathbb{C}} \text{Hom}_{\mathbb{C}}(G_\sigma, E_\sigma))^{F_\infty}$. Moreover its class $[(T_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}}]$ modulo $\text{Hom}_{\mathcal{O}_K}(G, E)$ in the extension group $\widehat{\text{Ext}}_{\mathcal{O}_K}^1(G, E)$ defined by (2.14) does not depend on the choice of the integral splitting s^{int} . By definition, it is the class $[(\mathcal{E}, (s(\sigma))_{\sigma:K \hookrightarrow \mathbb{C}})]$ of the arithmetic extension $(\mathcal{E}, (s(\sigma))_{\sigma:K \hookrightarrow \mathbb{C}})$.

2.4.6. Admissible short exact sequences and arithmetic extensions. Any admissible short exact sequence of hermitian vector bundles over $\text{Spec } \mathcal{O}_K$

$$\overline{\mathcal{E}} : 0 \rightarrow \overline{E} \xrightarrow{i} \overline{F} \xrightarrow{p} \overline{G} \rightarrow 0$$

determines an arithmetic extension $(\mathcal{E}, (s_\sigma^\perp)_{\sigma:K \hookrightarrow \mathbb{C}})$ of G by E , defined by means of the orthogonal splittings with respect to the hermitian norms $\|\cdot\|_{\overline{F}, \sigma}$ of the extensions of complex vector spaces \mathcal{E}_σ .

According to the discussion in 2.4.4, its class

$$[\overline{\mathcal{E}}] := [(\mathcal{E}, (s_\sigma^\perp)_{\sigma:K \hookrightarrow \mathbb{C}})]$$

vanishes in $\widehat{\text{Ext}}_{\mathcal{O}_K}^1(G, E)$ if and only if the admissible short exact sequence $\widehat{\text{Ext}}_{\mathcal{O}_K}^1(G, E)$ is split.

Besides, for any two hermitian vector bundle \overline{E} and \overline{G} over $\text{Spec } \mathcal{O}_K$, any class in $\widehat{\text{Ext}}_{\mathcal{O}_K}^1(G, E)$ may be realized by the previous construction, starting from some suitable admissible short exact sequence \mathcal{E} as above.

Indeed, for any $T = (T(\sigma))_{\sigma:K \hookrightarrow \mathbb{C}}$ in $(\bigoplus_{\sigma:K \hookrightarrow \mathbb{C}} \text{Hom}_{\mathbb{C}}(G_\sigma, E_\sigma))^{F_\infty}$, we may introduce the hermitian vector bundle over $\text{Spec } \mathcal{O}_K$

$$\overline{E \oplus G}^T := (E \oplus G, (\|\cdot\|_{T(\sigma)})_{\sigma:K \hookrightarrow \mathbb{C}})$$

where, for any field embedding $\sigma : K \hookrightarrow \mathbb{C}$ and any $(e, g) \in E_\sigma \oplus G_\sigma$,

$$\|(e, g)\|_{T(\sigma)}^2 := \|e - T_\sigma(g)\|_{E_\sigma}^2 + \|g\|_{G_\sigma}^2.$$

One easily checks that the maps

$$\begin{aligned} i : E &\longrightarrow E \oplus G \\ e &\longmapsto (e, 0) \end{aligned}$$

and

$$\begin{aligned} p : E \oplus G &\longrightarrow G \\ (e, g) &\longmapsto g \end{aligned}$$

fit into an admissible short exact sequence of hermitian vector bundles over $\text{Spec } \mathcal{O}_K$

$$(2.15) \quad \overline{\mathcal{E}}(T) : 0 \rightarrow \overline{E} \xrightarrow{i} \overline{E \oplus G}^T \xrightarrow{p} \overline{G} \rightarrow 0,$$

and that the associated class in $\widehat{\text{Ext}}_{\mathcal{O}_K}^1(G, E)$ is $[T]$.

2.4.7. *Direct images of arithmetic and admissible extensions.* Consider two finitely generated projective \mathcal{O}_K -modules E and G and the underlying \mathbb{Z} -modules π_*E and π_*G .

There is an obvious inclusion map between modules of \mathcal{O}_K -linear and \mathbb{Z} -linear morphisms:

$$\mathrm{Hom}_{\mathcal{O}_K}(G, E) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(\pi_*G, \pi_*E).$$

We will sometimes denote by $\pi_*\phi$ the image in $\mathrm{Hom}_{\mathbb{Z}}(\pi_*G, \pi_*E)$ of some element ϕ of the group $\mathrm{Hom}_{\mathcal{O}_K}(G, E)$.

In this paragraph, we want to investigate the “derived” analogue of this map π_* , defined between the relevant arithmetic extension groups.

Consider an arithmetic extension (\mathcal{E}, s) of E by G over $\mathrm{Spec} \mathcal{O}_K$, defined by an extension

$$\mathcal{E} : 0 \longrightarrow E \xrightarrow{i} F \xrightarrow{p} G \longrightarrow 0,$$

of \mathcal{O}_K -modules and a family $s = (s_\sigma)_{\sigma: K \hookrightarrow \mathbb{C}}$, invariant under complex conjugation, of \mathbb{C} -linear splittings $s_\sigma : G_\sigma \longrightarrow F_\sigma$ of the extensions of complex vector spaces

$$\mathcal{E}_\sigma : 0 \longrightarrow E_\sigma \xrightarrow{i_\sigma} F_\sigma \xrightarrow{p_\sigma} G_\sigma \longrightarrow 0.$$

We may define its direct image by $\pi : \mathrm{Spec} \mathcal{O}_K \longrightarrow \mathrm{Spec} \mathbb{Z}$ as the arithmetic extension $(\pi_*\mathcal{E}, \pi_*s)$ of π_*E by π_*G over $\mathrm{Spec} \mathbb{Z}$ defined by the extension of \mathbb{Z} -modules

$$\pi_*\mathcal{E} : 0 \longrightarrow \pi_*E \xrightarrow{i} \pi_*F \xrightarrow{p} \pi_*G \longrightarrow 0,$$

equipped with the splitting

$$\begin{aligned} \pi_*s : G \otimes_{\mathbb{Z}} \mathbb{C} &\simeq \bigoplus_{\sigma: K \hookrightarrow \mathbb{C}} G_\sigma \longrightarrow E \otimes_{\mathbb{Z}} \mathbb{C} \simeq \bigoplus_{\sigma: K \hookrightarrow \mathbb{C}} E_\sigma \\ (g_\sigma)_{\sigma: K \hookrightarrow \mathbb{C}} &\longmapsto (s_\sigma(g_\sigma))_{\sigma: K \hookrightarrow \mathbb{C}}. \end{aligned}$$

Proposition 2.4.1. 1) *The above construction defines a morphism of \mathbb{Z} -modules:*

$$\begin{aligned} \hat{\pi}_*^1 : \widehat{\mathrm{Ext}}_{\mathcal{O}_K}^1(G, E) &\longrightarrow \widehat{\mathrm{Ext}}_{\mathbb{Z}}^1(\pi_*G, \pi_*E) \\ [(\mathcal{E}, s)] &\longmapsto [(\pi_*\mathcal{E}, \pi_*s)]. \end{aligned}$$

2) *Via the identifications*

$$\widehat{\mathrm{Ext}}_{\mathcal{O}_K}^1(G, E) \xrightarrow{\sim} \frac{(\bigoplus_{\sigma: K \hookrightarrow \mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(G_\sigma, E_\sigma))^{F_\infty}}{\mathrm{Hom}_{\mathcal{O}_K}(G, E)}$$

and

$$\widehat{\mathrm{Ext}}_{\mathbb{Z}}^1(\pi_*G, \pi_*E) \xrightarrow{\sim} \frac{\mathrm{Hom}_{\mathbb{C}}(\pi_*G \otimes_{\mathbb{Z}} \mathbb{C}, \pi_*E \otimes_{\mathbb{Z}} \mathbb{C})^{F_\infty}}{\mathrm{Hom}_{\mathbb{Z}}(\pi_*G, \pi_*E)},$$

the map $\hat{\pi}_*^1$ coincides with the map defined, for any $(T(\sigma))_{\sigma: K \hookrightarrow \mathbb{C}}$ in $(\bigoplus_{\sigma: K \hookrightarrow \mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(G_\sigma, E_\sigma))^{F_\infty}$, by

$$\hat{\pi}_*^1[(T(\sigma))_{\sigma: K \hookrightarrow \mathbb{C}}] = [\bigoplus_{\sigma} T(\sigma)],$$

where $\bigoplus_{\sigma} T(\sigma)$ is the element of $\mathrm{Hom}_{\mathbb{C}}(\pi_*G \otimes_{\mathbb{Z}} \mathbb{C}, \pi_*E \otimes_{\mathbb{Z}} \mathbb{C})^{F_\infty}$ defined by

$$\begin{aligned} \bigoplus_{\sigma} T(\sigma) : \pi_*G \otimes_{\mathbb{Z}} \mathbb{C} &\simeq \bigoplus_{\sigma: K \hookrightarrow \mathbb{C}} G_\sigma \longrightarrow \pi_*E \otimes_{\mathbb{Z}} \mathbb{C} \simeq \bigoplus_{\sigma: K \hookrightarrow \mathbb{C}} E_\sigma \\ (g_\sigma)_{\sigma: K \hookrightarrow \mathbb{C}} &\longmapsto (T_\sigma(g_\sigma))_{\sigma: K \hookrightarrow \mathbb{C}}. \end{aligned}$$

3) *Any admissible short exact sequence of hermitian vector bundles over $\mathrm{Spec} \mathcal{O}_K$*

$$\overline{\mathcal{E}} : 0 \longrightarrow \overline{E} \xrightarrow{i} \overline{F} \xrightarrow{p} \overline{G} \longrightarrow 0$$

defines, by direct image, an admissible short exact sequence of hermitian vector bundles over $\mathrm{Spec} \mathbb{Z}$:

$$\pi_*\overline{\mathcal{E}} : 0 \longrightarrow \pi_*\overline{E} \xrightarrow{i} \pi_*\overline{F} \xrightarrow{p} \pi_*\overline{G} \longrightarrow 0.$$

Moreover this construction is compatible with the direct image map $\hat{\pi}_*^1$ between arithmetic extension groups. Namely, with the above notation, the following equality holds in $\widehat{\text{Ext}}_{\mathbb{Z}}^1(\pi_*G, \pi_*E)$:

$$\hat{\pi}_*^1[\bar{\mathcal{E}}] = [\pi_*\bar{\mathcal{E}}].$$

4) The map $\hat{\pi}_*^1$ is injective.

Proof. Assertions 1), 2), and 3) are left as an easy exercise for the reader.

To prove the injectivity of $\hat{\pi}_*^1$, we use its description in 2), and we are left to show that, with the notation of 2), if $\oplus_{\sigma} T(\sigma)$ belongs to $\text{Hom}_{\mathbb{Z}}(\pi_*G, \pi_*E)$, then there exists T in $\text{Hom}_{\mathcal{O}_K}(G, E)$ such that, for every $\sigma : K \hookrightarrow \mathbb{C}$, $T(\sigma) = T_{\sigma}$.

In other words, we have to show that, if some U in $\text{Hom}_{\mathbb{Z}}(\pi_*G, \pi_*E)$ is such that

$$U_{\mathbb{C}} : \pi_*G \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow \pi_*E \otimes_{\mathbb{Z}} \mathbb{C}$$

is block-diagonal with respect to the decompositions in direct sums

$$\pi_*G \otimes_{\mathbb{Z}} \mathbb{C} \simeq \bigoplus_{\sigma: K \hookrightarrow \mathbb{C}} G_{\sigma} \text{ and } \pi_*E \otimes_{\mathbb{Z}} \mathbb{C} \simeq \bigoplus_{\sigma: K \hookrightarrow \mathbb{C}} E_{\sigma},$$

then $U : G \longrightarrow E$ is \mathcal{O}_K -linear. This is indeed the case, since any such U is \mathcal{O}_K -linear if and only if $U_{\mathbb{C}} := U \otimes_{\mathbb{Z}} \text{Id}_{\mathbb{C}}$ is linear as a map of modules over the ring $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{C} \simeq \bigoplus_{\sigma: K \hookrightarrow \mathbb{C}} \mathbb{C}$. \square

3. θ -INVARIANTS OF HERMITIAN VECTOR BUNDLES OVER ARITHMETIC CURVES

This section is devoted to the definitions and to some basic properties of the θ -invariants of (finite-rank) hermitian vector bundles over arithmetic curves. Their extensions to infinite-dimensional hermitian vector bundles established in the latter sections of this article, as well as their Diophantine applications discussed in its sequel, will depend on these properties.

As explained in the Introduction, the definitions of the invariants h_{θ}^0 and h_{θ}^1 are implicit in the classical works of Hecke and F. K. Schmidt. Formal definitions and some of their basic properties already appear in [Roe93, 6.2], [Mor95, 1.4], and, most importantly from the perspective of this article, in the work of van der Geer and Schoof [vdGS00] and of Groenewegen [Gro01].

In this section, we give a self-contained and streamlined presentation of the theory of θ -invariants, adapted to our later use of them. We notably emphasize the importance of their subadditivity properties (see notably Lemma 3.8.2 and Section 4.3), which play a key role in the proof of the main results of this article in Section 8.

Among the results in this part that, by their novelty, might be of special interest to readers already familiar with the content of [vdGS00] and [Gro01], we mention:

- Proposition 3.3.3 and its corollaries, that provide useful comparison estimates for the θ -invariants of euclidean lattices with the same underlying \mathbb{Q} -vector spaces.
- The improved bounds on the invariant $h_{\theta}^0(\bar{L})$ of an hermitian line bundle \bar{L} over an arithmetic curve. These bounds turns out to be remarkably similar to the well-known bounds on the dimension $h^0(C, L)$ of the space of sections of a line bundle L over a projective curve C (see Propositions 3.7.2 and 3.7.3).
- The discussion of the subadditivity properties of h_{θ}^0 and h_{θ}^1 in Section 3.8, notably the results concerning the “subadditivity measure” $h_{\theta}(\bar{\mathcal{E}})$ attached to an admissible short exact sequence of euclidean lattices $\bar{\mathcal{E}}$ that are established in Propositions 3.8.3 and 3.8.4.

As in Section 2, we denote by K some number field, by \mathcal{O}_K its ring of integers and by π the morphism of schemes from $\text{Spec } \mathcal{O}_K$ to $\text{Spec } \mathbb{Z}$.

3.1. The Poisson formula. We first review some basic results related to the Poisson formula. This gives us the opportunity to introduce various conventions and notation that will be used in the next sections.

3.1.1. The Poisson formula for Schwartz functions. Let \overline{V} be a hermitian vector bundle over $\text{Spec } \mathbb{Z}$ (or, equivalently, a euclidean lattice). We shall denote $\lambda_{\overline{V}}$ the Lebesgue measure on $V_{\mathbb{R}}$. It is the unique translation invariant Radon measure on $V_{\mathbb{R}}$ that satisfies the following normalization condition: for any orthonormal basis (e_1, \dots, e_N) of $(V_{\mathbb{R}}, \|\cdot\|_{\overline{V}})$,

$$\lambda_{\overline{V}} \left(\sum_{i=1}^N [0, 1[e_1] \right) = 1.$$

This normalization condition may be equivalently expressed in term of a Gaussian integral:

$$(3.1) \quad \int_{V_{\mathbb{R}}} e^{-\pi \|x\|_{\overline{V}}^2} d\lambda_{\overline{V}}(x) = 1.$$

Then the covolume of \overline{V} may be defined as

$$\text{covol}(\overline{V}) := \lambda_{\overline{V}} \left(\sum_{i=1}^N [0, 1[v_i] \right)$$

for any \mathbb{Z} -basis (v_1, \dots, v_N) of V .

Any function f in the Schwartz space $\mathcal{S}(V_{\mathbb{R}})$ of $V_{\mathbb{R}}$ admits a Fourier transform \hat{f} in the Schwartz space $\mathcal{S}(V_{\mathbb{R}}^{\vee})$ attached to the dual vector space $V_{\mathbb{R}}^{\vee}$, defined by

$$\hat{f}(\xi) := \int_{V_{\mathbb{R}}} f(x) e^{-2\pi i \langle \xi, x \rangle} d\lambda_{\overline{V}}(x), \text{ for any } \xi \in V_{\mathbb{R}}^{\vee}.$$

With this notation, the Poisson formula for \overline{V} reads as follows:

$$(3.2) \quad \sum_{v \in V} f(x - v) = (\text{covol} \overline{V})^{-1} \sum_{v^{\vee} \in V^{\vee}} \hat{f}(v^{\vee}) e^{2\pi i \langle v^{\vee}, x \rangle}, \text{ for any } x \in V_{\mathbb{R}}.$$

It is nothing but the Fourier series expansion of the function $\sum_{v \in V} f(\cdot - v)$, which is V -periodic on $V_{\mathbb{R}}$.

It is convenient to have at one's disposal the following more “symmetric” form of the Poisson formula:

$$(3.3) \quad \sum_{v \in V} f(x - v) e^{2\pi i \langle \xi, v \rangle} = (\text{covol} \overline{V})^{-1} e^{2\pi i \langle \xi, x \rangle} \sum_{v^{\vee} \in V^{\vee}} \hat{f}(\xi - v^{\vee}) e^{-2\pi i \langle v^{\vee}, x \rangle},$$

which is valid for any $(x, \xi) \in V_{\mathbb{R}} \times V_{\mathbb{R}}^{\vee}$. (The identity (3.3) indeed follows from (3.2) applied to the function $e^{-2\pi i \langle \xi, \cdot \rangle} f(\cdot)$.)

3.1.2. The Poisson formula for Gaussian functions. In this article, a key role will be played by the Poisson formula applied to Gaussian functions. Namely, we apply the above formula to the Gaussian function f defined by

$$f(x) := e^{-\pi \|x\|_{\overline{V}}^2}.$$

Then, for any $\xi \in V_{\mathbb{R}}^{\vee}$, we have:

$$\hat{f}(\xi) = e^{-\pi \|\xi\|_{\overline{V}^{\vee}}^2},$$

and the identity (3.2) and (3.3) take the form:

$$(3.4) \quad \sum_{v \in V} e^{-\pi \|x - v\|_{\overline{V}}^2} = (\text{covol} \overline{V})^{-1} \sum_{v^{\vee} \in V^{\vee}} e^{-\pi \|v^{\vee}\|_{\overline{V}^{\vee}}^2 + 2\pi i \langle v^{\vee}, x \rangle}$$

and

$$(3.5) \quad \sum_{v \in V} e^{-\pi \|x-v\|_{\overline{V}}^2 + 2\pi i \langle \xi, v \rangle} = (\text{covol} \overline{V})^{-1} e^{2\pi i \langle \xi, x \rangle} \sum_{v^\vee \in V^\vee} e^{-\pi \|\xi - v^\vee\|_{\overline{V}^\vee}^2 - 2\pi i \langle v^\vee, x \rangle}$$

In particular, when $x = 0$, the identity (3.4) becomes:

$$(3.6) \quad \sum_{v \in V} e^{-\pi \|v\|_{\overline{V}}^2} = (\text{covol} \overline{V})^{-1} \sum_{v^\vee \in V^\vee} e^{-\pi \|v^\vee\|_{\overline{V}^\vee}^2}.$$

Observe also that (3.4) implies that, for any $x \in V_{\mathbb{R}}$,

$$(3.7) \quad \sum_{v \in V} e^{-\pi \|x-v\|_{\overline{V}}^2} \leq \sum_{v \in V} e^{-\pi \|v\|_{\overline{V}}^2},$$

and that equality holds in (3.7) if and only if $x \in V$. It also implies that, for any $x \in V_{\mathbb{R}}$,

$$(3.8) \quad \sum_{v \in V} e^{-\pi \|x-v\|_{\overline{V}}^2} \geq 2(\text{covol} \overline{V})^{-1} - \sum_{v \in V} e^{-\pi \|v\|_{\overline{V}}^2}.$$

3.2. The θ -invariants h_θ^0 and h_θ^1 and the Poisson-Riemann-Roch formula.

3.2.1. *The θ -invariants of an euclidean lattice and the Poisson formula.* For any hermitian vector bundle $\overline{E} := (E, \|\cdot\|)$ over $\text{Spec } \mathbb{Z}$ (that is, for any euclidean lattice), we let :

$$(3.9) \quad h_\theta^0(\overline{E}) := \log \sum_{v \in E} e^{-\pi \|v\|_{\overline{E}}^2}$$

and

$$(3.10) \quad h_\theta^1(\overline{E}) := h_\theta^0(\overline{E}^\vee).$$

The Poisson formula (3.6) for the euclidean lattice \overline{E} reads:

$$\sum_{w \in E^\vee} e^{-\pi \|w\|_{\overline{E}^\vee}^2} = \text{covol}(\overline{E}). \sum_{v \in E} e^{-\pi \|v\|_{\overline{E}}^2}.$$

It may be rewritten in terms of the θ -invariants $h_\theta^0(\overline{E})$ and $h_\theta^1(\overline{E})$ and of the Arakelov degree $\widehat{\deg} \overline{E}$, as the following relation:

$$(3.11) \quad h_\theta^0(\overline{E}) - h_\theta^1(\overline{E}) = \widehat{\deg} \overline{E}.$$

Observe the similarity of (3.10) and (3.11) with the Serre duality and the Riemann-Roch formula for vector bundles over an elliptic curve.

3.2.2. *The θ -invariants of hermitian vector bundles over a general arithmetic curve and the Poisson-Riemann-Roch formula.* We extend the above definitions of h_θ^0 and h_θ^1 to hermitian vector bundles over an arbitrary “arithmetic curve” $\text{Spec } \mathcal{O}_K$ as above by defining:

$$(3.12) \quad h_\theta^i(\overline{E}) := h_\theta^i(\pi_* \overline{E}) \text{ for } i = 0, 1.$$

In other words, we have:

$$h_\theta^0(\overline{E}) = \log \sum_{v \in E} e^{-\pi \|v\|_{\pi_* \overline{E}}^2}$$

and

$$h_\theta^1(\overline{E}) = h_\theta^0(\overline{E}) - \widehat{\deg} \pi_* \overline{E} = \log \left[\text{covol}(\pi_* \overline{E}) \sum_{v \in E} e^{-\pi \|v\|_{\pi_* \overline{E}}^2} \right].$$

Observe that, as a consequence of the “relative duality isomorphism” (2.1), we have:

$$(3.13) \quad h_\theta^1(\overline{E}) = h_\theta^0(\overline{E}^\vee \otimes \overline{\omega}_{\mathcal{O}_K/\mathbb{Z}}).$$

This “Serre duality formula” could have been used as an alternative definition of $h_\theta^1(\overline{E})$.

Observe finally that, as a consequence of the Poisson formula (3.11) for euclidean lattices and of the expression (2.7) for the Arakelov degree of a direct image, we obtain the general version of the *Poisson-Riemann-Roch formula*, where we denote by Δ_K the discriminant of the number field K :

$$(3.14) \quad h_\theta^0(\overline{E}) - h_\theta^1(\overline{E}) = \widehat{\deg} \overline{E} - \frac{1}{2} \log |\Delta_K| \cdot \text{rk } E.$$

3.3. Positivity and monotonicity.

3.3.1. Some elementary estimates. The following observation is a straightforward consequence from the definitions (3.9) and (3.10) of the θ -invariants, but plays an important conceptual role in this article and its applications:

Proposition 3.3.1. *For any hermitian vector bundle \overline{E} over $\text{Spec } \mathcal{O}_K$, the real numbers $h_\theta^0(\overline{E})$ and $h_\theta^1(\overline{E})$ are non-negative, and are positive if \overline{E} has positive rank. \square*

Together with the “Poisson-Riemann-Roch formula” (3.14), this non-negativity shows that, for any hermitian vector bundle, the following avatar of the “Riemann inequality” is satisfied:

$$h_\theta^0(\overline{E}) \geq \widehat{\deg} \overline{E} - \frac{1}{2} \log |\Delta_K| \cdot \text{rk } E.$$

In particular, for any euclidean lattice \overline{E} , the following lower bound is satisfied:

$$(3.15) \quad h_\theta^0(\overline{E}) \geq \widehat{\deg} \overline{E}.$$

Proposition 3.3.2. *Let \overline{E} and \overline{F} be two hermitian vector bundles over $\text{Spec } \mathcal{O}_K$, and let $\phi : E \rightarrow F$ be a map in $\text{Hom}_{\mathcal{O}_K}^{\leq 1}(\overline{E}, \overline{F})$. Let $\phi_K : E_K \rightarrow F_K$ denote the induced morphism of K -vector spaces.*

1) If ϕ_K is injective (or equivalently, if ϕ is injective), then

$$(3.16) \quad h_\theta^0(\overline{E}) \leq h_\theta^0(\overline{F}).$$

2) If ϕ_K is surjective, then

$$(3.17) \quad h_\theta^1(\overline{E}) \geq h_\theta^1(\overline{F}).$$

Moreover, equality holds in either (3.16) or (3.17) if and only if ϕ is an isometric isomorphism from \overline{E} onto \overline{F} .

Proof. 1) Let us assume that ϕ_K is injective.

By successively using that the maps ϕ_σ have operator norm at most 1 and the injectivity of ϕ , we obtain:

$$(3.18) \quad \sum_{v \in E} e^{-\pi \|v\|_{\pi_* \overline{E}}^2} \leq \sum_{v \in E} e^{-\pi \|\phi(v)\|_{\pi_* \overline{F}}^2}$$

$$(3.19) \quad \leq \sum_{w \in F} e^{-\pi \|w\|_{\pi_* \overline{F}}^2}.$$

This establishes (3.16).

Equality holds in (3.18) if and only if for every field embedding $\sigma : K \hookrightarrow \mathbb{C}$ and every $v \in E$,

$$\|\phi_\sigma(v)\|_{\overline{F}, \sigma}^2 = \|v\|_{\overline{E}, \sigma}^2.$$

Since the family of hermitian norms $(\|\cdot\|_{\overline{E}, \sigma})_{\sigma: K \hookrightarrow \mathbb{C}}$ and $(\|\cdot\|_{\overline{F}, \sigma})_{\sigma: K \hookrightarrow \mathbb{C}}$ are invariant under complex conjugation, this holds precisely when the ϕ_σ are isometries.

Moreover equality holds in (3.19) if and only if $\phi(E) = F$.

This shows that equality holds in (3.16) if and only if ϕ is an isometric isomorphism.

2) When ϕ_K is surjective, we consider the morphism

$$\pi_* \phi \in \text{Hom}_{\mathbb{Z}}^{\leq 1}(\pi_* \overline{E}, \pi_* \overline{F})$$

and its transpose

$${}^t(\pi_*\phi) \in \text{Hom}_{\overline{\mathbb{Z}}}^{\leq 1}((\pi_*\overline{F})^\vee, (\pi_*\overline{E})^\vee).$$

It is injective and, according to 1),

$$(3.20) \quad h_\theta^0((\pi_*\overline{F})^\vee) \leq h_\theta^0((\pi_*\overline{E})^\vee).$$

This establishes (3.17).

Moreover equality holds in (3.17), or equivalently in (3.20), if and only if ${}^t(\pi_*\phi)$ is an isometric isomorphism from $(\pi_*\overline{F})^\vee$ onto $(\pi_*\overline{E})^\vee$. This is easily seen to be equivalent to the fact that ϕ itself is an isometric isomorphism. \square

3.3.2. Comparing the θ -invariants of generically isomorphic hermitian vector bundles. Combined to the Poisson-Riemann-Roch formula and to the basic properties of the height of morphisms (cf. 2.4.3, *supra*), the simple estimates established in Proposition 3.3.2 may be extended to more general situations where one deals with two hermitian vector bundles \overline{E} and \overline{F} over $\text{Spec } \mathcal{O}_K$ such that E_K and F_K are isomorphic

Proposition 3.3.3. *Let \overline{E} and \overline{F} be two hermitian vector bundles over $\text{Spec } \mathcal{O}_K$ of the same rank n and let $\phi : E \rightarrow F$ be a map in $\text{Hom}_{\mathcal{O}_K}^{\leq 1}(\overline{E}, \overline{F})$.*

If ϕ is injective, or equivalently if

$$\det \phi_K : \wedge^n E_K \rightarrow \wedge^n F_K$$

is not zero, then the following inequalities hold:

$$(3.21) \quad h_\theta^0(\overline{E}) \leq h_\theta^0(\overline{F}) \leq h_\theta^0(\overline{E}) - \text{ht}(\wedge^n \overline{E}, \wedge^n \overline{F}, \det \phi_K).$$

and

$$h_\theta^1(\overline{F}) \leq h_\theta^1(\overline{E}) \leq h_\theta^1(\overline{F}) - \text{ht}(\wedge^n \overline{E}, \wedge^n \overline{F}, \det \phi_K).$$

Proof. The inequalities

$$(3.22) \quad h_\theta^0(\overline{E}) \leq h_\theta^0(\overline{F})$$

and

$$(3.23) \quad h_\theta^1(\overline{F}) \leq h_\theta^1(\overline{E})$$

are special cases of Proposition 3.3.2.

Besides, according to (2.10),

$$\text{ht}(\wedge^n \overline{E}, \wedge^n \overline{F}, \det \phi_K) = \widehat{\deg} \wedge^n \overline{E} - \widehat{\deg} \wedge^n \overline{F} = \widehat{\deg} \overline{E} - \widehat{\deg} \overline{F}.$$

The inequality

$$h_\theta^0(\overline{F}) \leq h_\theta^0(\overline{E}) - \text{ht}(\wedge^n \overline{E}, \wedge^n \overline{F}, \det \phi_K)$$

may therefore be written

$$h_\theta^0(\overline{F}) - \widehat{\deg} \overline{F} \leq h_\theta^0(\overline{E}) - \widehat{\deg} \overline{E}.$$

Taking the Poisson-Riemann-Roch formula (3.14) for \overline{E} and \overline{F} into account, it reduces to (3.23).

Similarly, the inequality

$$h_\theta^1(\overline{E}) \leq h_\theta^1(\overline{F}) - \text{ht}(\wedge^n \overline{E}, \wedge^n \overline{F}, \det \phi_K)$$

follows from (3.22). \square

Corollary 3.3.4. *For any hermitian vector bundle $\overline{F} := (F, (\|\cdot\|_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}})$ over $\text{Spec } \mathcal{O}_K$ and every \mathcal{O}_K -submodule E of F such that $E_K = F_K$ — or equivalently, such that the quotient F/E is finite — the θ -invariants of the hermitian vector bundle $\overline{E} := (E, (\|\cdot\|_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}})$ satisfy:*

$$h_\theta^0(\overline{F}) - \log |F/E| \leq h_\theta^0(\overline{E}) \leq h_\theta^0(\overline{F})$$

and

$$h_\theta^1(\overline{F}) \leq h_\theta^1(\overline{E}) \leq h_\theta^1(\overline{F}) + \log |F/E|.$$

Proof. Indeed, if $\phi : E \rightarrow F$ denotes the inclusion morphism, we have:

$$-\text{ht}(\wedge^n \overline{E}, \wedge^n \overline{F}, \det \phi_K) = -\widehat{\deg} \overline{E} + \widehat{\deg} \overline{F} = \log |F/E|.$$

□

Recall that, by definition, an *Arakelov divisor*

$$\overline{D} := \left(\sum_i n_i \mathfrak{p}_i, (\delta_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}} \right)$$

over $\text{Spec } \mathcal{O}_K$ consists in a divisor $\sum_i n_i \mathfrak{p}_i$ in $\text{Spec } \mathcal{O}_K$ and in a family, invariant under complex conjugation, of $[K : \mathbb{Q}]$ real numbers $(\delta_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}}$. It is *effective* if the divisor D and the δ_σ 's are non-negative.

To \overline{D} is attached a hermitian line bundle

$$\mathcal{O}(\overline{D}) := (\mathcal{O}_K(D), (\|\cdot\|_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}})$$

where $\mathcal{O}_K(D) := \prod_i \mathfrak{p}_i^{-n_i}$ and where the metric $\|\cdot\|_\sigma$ on $\mathcal{O}_K(D)_\sigma \simeq \mathbb{C}$ is defined by $\|1\|_\sigma := e^{-\delta_\sigma}$. The Arakelov degree of \overline{D} is defined as:

$$\widehat{\deg} \overline{D} := \widehat{\deg} \mathcal{O}(\overline{D}) = \sum_i n_i \log N \mathfrak{p}_i + \sum_\sigma \delta_\sigma.$$

Corollary 3.3.5. *Let \overline{E} be a hermitian vector bundle over $\text{Spec } \mathcal{O}_K$.*

1) *For any effective Arakelov divisor \overline{D} over $\text{Spec } \mathcal{O}_K$, we have*

$$(3.24) \quad 0 \leq h_\theta^0(\overline{E}) - h_\theta^0(\overline{E} \otimes \mathcal{O}(-\overline{D})) \leq \text{rk } E \cdot \widehat{\deg} \overline{D}.$$

2) *For any δ in \mathbb{R}_+ ,*

$$(3.25) \quad 0 \leq h_\theta^0(\overline{E}) - h_\theta^0(\overline{E} \otimes \overline{\mathcal{O}}(-\delta)) \leq \text{rk } E \cdot [K : \mathbb{Q}] \cdot \delta.$$

Proof. 1) Let

$$\overline{E}' := \overline{E} \otimes \mathcal{O}(-\overline{D}).$$

As \overline{D} is effective, the module

$$E' = E \otimes \mathcal{O}_K(-D) = \prod_i \mathfrak{p}_i^{n_i} E$$

is a \mathcal{O}_K -submodule of E , of the same rank $\text{rk } E$ as E , and the inclusion morphism $\phi : E' \hookrightarrow E$ has hermitian operator norms $\|\phi_\sigma\| = e^{-\delta_\sigma} \leq 1$.

Moreover, the norms of $\det \phi_K$ are easily computed:

$$\begin{aligned} \|\det \phi_K\|_{\mathfrak{p}} &= 1 \text{ if } \mathfrak{p} \notin \{\mathfrak{p}_i\}_i \\ &= (N \mathfrak{p}_i)^{-\text{rk } E \cdot n_i} \text{ if } \mathfrak{p} = \mathfrak{p}_i. \end{aligned}$$

and, for every embedding $\sigma : K \hookrightarrow \mathbb{C}$,

$$\|\det \phi_K\|_\sigma = e^{-\text{rk } E \cdot \delta_\sigma}.$$

Therefore

$$h(\wedge^{\mathrm{rk} E} \overline{E}', \wedge^{\mathrm{rk} E} \overline{E}, \det \phi_K) = -\mathrm{rk} E \left(\sum_i n_i \log N \mathfrak{p}_i + \sum_{\sigma: K \hookrightarrow \mathbb{C}} \delta_\sigma \right) = -\mathrm{rk} E \cdot \widehat{\deg} \overline{D}.$$

and (3.24) follows from (3.21) applied to the morphism ϕ in $\mathrm{Hom}_{\overline{\mathcal{O}}_K}^{\leq 1}(\overline{E}', \overline{E})$.

2) The hermitian line bundle $\overline{\mathcal{O}}_{\mathrm{Spec} \mathcal{O}_K}(\delta)$ is of the form $\mathcal{O}(\overline{D})$ for some effective Arakelov divisor \overline{D} of Arakelov degree $\widehat{\deg} \overline{D} = [K : \mathbb{Q}] \cdot \delta$ (namely $\overline{D} := (0, (\delta_\sigma)_{\sigma: K \hookrightarrow \mathbb{C}}$ where $\delta_\sigma := \delta$ for every $\sigma : K \hookrightarrow \mathbb{C}$). Therefore (3.25) follows from (3.24). \square

3.4. The functions τ and η . To express the θ -invariants of euclidean lattices of rank one, it will be convenient to introduce the two functions

$$\tau : \mathbb{R}_+^* \longrightarrow \mathbb{R}_+^* \text{ and } \eta : \mathbb{R} \longrightarrow \mathbb{R}_+^*$$

defined as follows.

For any $x \in \mathbb{R}_+^*$, we let:

$$\tau(x) := \log \sum_{n \in \mathbb{Z}} e^{-\pi x n^2}.$$

This definition may also be written:

$$\tau(x) = h_\theta^0(\overline{\mathcal{O}}(-\frac{1}{2} \log x))$$

and that the Poisson formula (3.6) (or equivalently the Poisson-Riemann-formula (3.11)) applied to $\overline{\mathcal{O}}(-\frac{1}{2} \log x)$ becomes:

$$(3.26) \quad \tau(x) = \tau(x^{-1}) - \frac{1}{2} \log x.$$

Observe also that:

$$(3.27) \quad \tau(x) = 2e^{-\pi x} + O(e^{-2\pi x}) \text{ when } x \longrightarrow +\infty.$$

Together with the Poisson formula (3.26), this implies:

$$(3.28) \quad \tau(x) = -\frac{1}{2} \log x + 2e^{-\pi/x} + O(e^{-2\pi/x}) \text{ when } x \longrightarrow 0_+.$$

For any $t \in \mathbb{R}$, we also let

$$\eta(t) := \tau(e^{2|t|}).$$

It is a continuous non-increasing function of $|t|$. In particular, its maximal value is

$$\max_{t \in \mathbb{R}} \eta(t) = \eta(0) = \tau(1).$$

We shall denote this number by η . In other words,

$$\eta = h_\theta^0(\overline{\mathcal{O}}) = \log \omega,$$

where

$$\omega := \sum_{n \in \mathbb{Z}} e^{-\pi n^2}.$$

The number η is denoted by $\eta(\mathbb{Q})$ in [vdGS00]. As explained in *loc. cit.*, proof of Proposition 4, the positive real number ω may be expressed as

$$\omega := \frac{\pi^{1/4}}{\Gamma(3/4)}.$$

Numerically, one finds:

$$\omega = 1.0864348... \text{ and } \eta = 0.0829015....$$

Moreover, according to (3.26), we have:

$$(3.29) \quad \eta(t) := \tau(e^{-2t}) - t^+,$$

or equivalently:

$$h_\theta^0(\overline{\mathcal{O}}(t)) = t^+ + \eta(t).$$

Actually, when $|t|$ goes to $+\infty$, $\eta(t)$ converges very fast to zero. Indeed, according to (3.27,) we have:

$$(3.30) \quad \eta(t) = 2e^{-\pi e^{2|t|}} + O(e^{-2\pi e^{2|t|}}) \text{ when } |t| \longrightarrow +\infty.$$

Observe that this implies the existence of some constant $c \in \mathbb{R}_+^*$ such that, for any $t \in \mathbb{R}$,

$$(3.31) \quad \eta(t) \leq ce^{-\pi e^{2|t|}},$$

or equivalently, such that for every $x \in [1, +\infty[$

$$(3.32) \quad \tau(x) \leq ce^{-\pi x}.$$

An elementary computation shows that this holds as soon as

$$c \geq 2(1 + \frac{e^{-3\pi}}{1 - e^{-5\pi}}) = 2.0002...$$

For instance, we may choose $c = 3$.

3.5. Additivity. The θ -invariants of direct sums of hermitian line bundles over $\text{Spec } \mathbb{Z}$. As a direct consequence of their definition, the invariants h_θ^0 and h_θ^1 also satisfy the following additivity property:

Proposition 3.5.1. *For any two hermitian vector bundles \overline{E} and \overline{F} over $\text{Spec } \mathcal{O}_K$,*

$$(3.33) \quad h_\theta^i(\overline{E} \oplus \overline{F}) = h_\theta^i(\overline{E}) + h_\theta^i(\overline{F}), \quad i = 0, 1.$$

□

If n denotes a positive integer and $\boldsymbol{\lambda} := (\lambda_1, \dots, \lambda_n)$ an element of \mathbb{R}_+^{*n} , we may consider the hermitian vector bundle \overline{V}_λ over $\text{Spec } \mathbb{Z}$ attached to the lattice \mathbb{Z}^n in \mathbb{R}^n equipped with the euclidean norm $\|\cdot\|_\lambda$ defined by

$$\|(x_1, \dots, x_n)\|_\lambda^2 := \sum_{i=1}^n \lambda_i x_i^2.$$

Clearly:

$$\overline{V}_\lambda \simeq \bigoplus_{1 \leq i \leq n} \overline{\mathcal{O}}(-\frac{1}{2} \log \lambda_i).$$

The Arakelov degree and the θ -invariants of V_λ are easily computed, by using their additivity (2.12) and (3.33) and the Poisson-Riemann-Roch formula (3.11):

Proposition 3.5.2. *For any positive integer n and any $\boldsymbol{\lambda} \in \mathbb{R}_+^{*n}$, we have:*

$$\widehat{\deg} \overline{V}_\lambda = -\frac{1}{2} \sum_{i=1}^n \log \lambda_i, \quad h_\theta^0(\overline{V}_\lambda) = \sum_{i=1}^n \tau(\lambda_i), \quad \text{and} \quad h_\theta^1(\overline{V}_\lambda) = \sum_{i=1}^n (\tau(\lambda_i) + \frac{1}{2} \log \lambda_i).$$

□

These formula may be rewritten as follows, in terms of the function η :

Proposition 3.5.3. *For any positive integer n and any hermitian line bundles $\overline{L}_1, \dots, \overline{L}_n$ over $\text{Spec } \mathbb{Z}$, we have:*

$$(3.34) \quad \widehat{\deg} \left(\bigoplus_{i=1}^n \overline{L}_i \right) = \sum_{i=1}^n \widehat{\deg} \overline{L}_i,$$

$$h_{\theta}^0 \left(\bigoplus_{i=1}^n \overline{L}_i \right) = \sum_{i=1}^n (\widehat{\deg}^+ \overline{L}_i + \eta(\widehat{\deg} \overline{L}_i))$$

and

$$(3.35) \quad h_{\theta}^1 \left(\bigoplus_{i=1}^n \overline{L}_i \right) = \sum_{i=1}^n (\widehat{\deg}^- \overline{L}_i + \eta(\widehat{\deg} \overline{L}_i)).$$

□

We have denoted by $\widehat{\deg}^+ \overline{L}_i$ and $\widehat{\deg}^- \overline{L}_i$ the positive and negative parts of the real number $\widehat{\deg} \overline{L}_i$.

Observe the similarity of (3.34) and (3.35) with the expressions for the dimensions of the coherent cohomology groups of a direct sum of line bundles over an elliptic curve.

3.6. The theta function $\theta_{\overline{E}}$ and the first minimum $\lambda_1(\overline{E})$. In this section, we consider hermitian vector bundles over $\text{Spec } \mathbb{Z}$ — that is euclidean lattices — and we discuss some relations between their θ -invariants and the invariants classically associated to them in geometry of numbers, such as their first minimum or their number of lattice points in the unit ball.

Most of the contents of this section is due to Groenewegen ([Gro01]). We shall pursue our study of the relations between θ -invariants and more classical invariants attached to euclidean lattices in Section 4.2 below, where we shall reformulate Banaszczyk's results ([Ban93]). Notably, in 4.2.2, we shall compare Groenewegen's and Banaszczyk's results relating the first minimum of some euclidean lattice with its θ -invariants.

3.6.1. The theta function $\theta_{\overline{E}}$. For any hermitian vector bundle $\overline{E} := (E, \|\cdot\|)$ over $\text{Spec } \mathbb{Z}$, it will be convenient to consider the associated theta function, defined for any t in \mathbb{R}_+^* by

$$\theta_{\overline{E}}(t) := \sum_{v \in E} e^{-\pi t \|v\|^2}.$$

We have, by the very definition of $h_{\theta}^0(\overline{E})$:

$$h_{\theta}^0(\overline{E}) = \log \theta_{\overline{E}}(1).$$

Moreover, for any δ in \mathbb{R} , if we denote $\overline{\mathcal{O}}(\delta)$ the hermitian line bundle over $\text{Spec } \mathbb{Z}$ of Arakelov degree $\widehat{\deg} \overline{\mathcal{O}}(\delta) = \delta$ (introduced in 2.1.3, *supra*), then

$$\theta_{\overline{E} \otimes \overline{\mathcal{O}}(\delta)}(t) = \sum_{v \in E} e^{-\pi t e^{-2\delta} \|v\|^2} = \theta_{\overline{E}}(e^{-2\delta} t).$$

Consequently

$$h_{\theta}^0(\overline{E} \otimes \overline{\mathcal{O}}(\delta)) = \log \theta_{\overline{E}}(e^{-2\delta}),$$

and, for any $t \in \mathbb{R}_+^*$,

$$\log \theta_{\overline{E}}(t) = h^0(\overline{E} \otimes \overline{\mathcal{O}}(-(\log t)/2)).$$

In other words, $\log \theta_{\overline{E}}$ is a kind of “arithmetic Hilbert function” for the hermitian vector bundle \overline{E} over $\text{Spec } \mathbb{Z}$.

Observe also that the classical functional equation for the theta function may be written :

$$(3.36) \quad \theta_{\overline{E}}(t) = t^{-\frac{1}{2} \text{rk } E} (\text{covol } \overline{E})^{-1} \theta_{\overline{E}^{\vee}}(t^{-1}),$$

or equivalently,

$$(3.37) \quad \log \theta_{\overline{E}}(t) = -\frac{1}{2} \operatorname{rk} E \cdot \log t + \widehat{\deg} \overline{E} + \log \theta_{\overline{E}^\vee}(t^{-1}).$$

It is nothing but the Poisson-Riemann-Roch formula 3.11 for the hermitian vector bundle $\overline{E} \otimes \overline{\mathcal{O}}(-(\log t)/2)$.

The theta function $\theta_{\overline{E}}$ may be related to the “naive” counting function

$$N_{\overline{E}} : \mathbb{R}_+ \longrightarrow \mathbb{N}$$

which controls the number of points in the euclidean lattice \overline{E} in balls centered at the origin. Indeed, if we define, for every $x \in \mathbb{R}_+$,

$$N_{\overline{E}}(x) := |\{v \in E \mid \|v\| \leq x\}|,$$

then the theta function $\theta_{\overline{E}}$ is basically the Laplace transform of $N_{\overline{E}}(\sqrt{\cdot})$:

Proposition 3.6.1. *With the above notation, for every $t \in \mathbb{R}_+^*$, we have:*

$$\theta_{\overline{E}}(t) = \pi t \int_0^{+\infty} N_{\overline{E}}(\sqrt{x}) e^{-\pi t x} dx.$$

Proof. Observe that, for any $x \in \mathbb{R}_+$,

$$\sum_{v \in E} \mathbf{1}_{[\|v\|^2, +\infty[}(x) = N_{\overline{E}}(x).$$

Consequently we have:

$$\begin{aligned} \theta_{\overline{E}}(t) &:= \sum_{v \in E} e^{-\pi t \|v\|^2} \\ &= \sum_{v \in E} \pi t \int_0^{+\infty} \mathbf{1}_{[\|v\|^2, +\infty[}(x) e^{-\pi t x} dx \\ &= \pi t \int_0^{+\infty} N_{\overline{E}}(\sqrt{x}) e^{-\pi t x} dx. \end{aligned}$$

(The last equality follows, for instance, from Lebesgue monotone convergence theorem.) \square

3.6.2. First minima of euclidean lattices and θ -invariants. For any hermitian vector bundle of positive rank $\overline{E} := (E, \|\cdot\|)$ over $\operatorname{Spec} \mathbb{Z}$, we may consider the first of its successive minima:

$$\lambda_1(\overline{E}) := \min \{\|v\|, v \in E \setminus \{0\}\},$$

and the number of its “short vectors”:

$$\nu := |\{v \in E \mid \|v\| = \lambda_1(\overline{E})\}|.$$

These quantities are easily seen to control the asymptotic behaviour of $\theta_{\overline{E}}(t)$ when t goes to $+\infty$, or equivalently the one of $h_{\theta}^0(\overline{E} \otimes \overline{\mathcal{O}}(\delta))$ when δ goes to $-\infty$. Indeed, when t goes to $+\infty$,

$$\theta_{\overline{E}}(t) = 1 + \nu e^{-\pi \lambda_1(\overline{E})^2 t} + O(e^{-\pi \lambda'^2 t}),$$

for some $\lambda' > \lambda_1(\overline{E})$. Consequently, for some positive ϵ ,

$$\log \theta_{\overline{E}}(t) = \nu e^{-\pi \lambda_1(\overline{E})^2 t} (1 + O(e^{-\epsilon t})) \text{ when } t \text{ goes to } +\infty,$$

and

$$(3.38) \quad h_{\theta}^0(\overline{E} \otimes \overline{\mathcal{O}}(\delta)) = \nu e^{-\pi \lambda_1(\overline{E})^2 e^{-2\delta}} (1 + O(e^{-\epsilon e^{-2\delta}})) \text{ when } \delta \text{ goes to } -\infty.$$

It is actually possible to derive an explicit upper bound on $h_{\theta}^0(\overline{E})$ in terms of $\lambda_1(\overline{E})$, namely:

Proposition 3.6.2. *For any euclidean lattice \overline{E} of positive rank n and of first minimum $\lambda := \lambda_1(\overline{E})$, the following estimates hold:*

$$(3.39) \quad h_\theta^0(\overline{E}) \leq e^{h_\theta^0(\overline{E})} - 1 \leq C(n, \lambda),$$

where

$$C(n, \lambda) := 3^n (\pi \lambda^2)^{-n/2} \int_{\pi \lambda^2}^{+\infty} u^{n/2} e^{-u} du.$$

Moreover, if $\lambda > (n/2\pi)^{1/2}$,

$$(3.40) \quad C(n, \lambda) \leq 3^n \left(1 - \frac{n}{2\pi \lambda^2}\right)^{-1} e^{-\pi \lambda^2}.$$

This Proposition is a slightly improved version of Proposition 4.4 in [Gro01], and we will follow Groenewegen's arguments — based on the expression of the theta function $\theta_{\overline{E}}$ as a Laplace transform of $N_{\overline{E}}(\sqrt{\cdot})$ — with minor modifications.

Observe that the estimates (3.39) and (3.40) applied to $\overline{E} \otimes \overline{\mathcal{O}}(\delta)$, compared with the asymptotic expression (3.38) when δ goes to $-\infty$ show that $\nu \leq 3^n$. Actually, it is classically known¹² that $\nu \leq 2(2^n - 1)$.

Lemma 3.6.3 (cf. [Gro01], Lemma 4.2). *With the notation of Proposition 3.6.2, for any x in \mathbb{R}_+ , we have:*

$$(3.41) \quad N_{\overline{E}}(x) \leq \left(\frac{2x}{\lambda} + 1\right)^n.$$

Consequently, for any x in $[\lambda, +\infty[$,

$$(3.42) \quad N_{\overline{E}}(x) \leq \left(\frac{3x}{\lambda}\right)^n.$$

Proof. For any $P \in E_{\mathbb{R}}$ and any $r \in \mathbb{R}_+$, let us denote by $\mathring{B}(P, r)$ the open ball of center P and radius r in the normed vector space $(E_{\mathbb{R}}, \|\cdot\|)$.

For any two points v and w of E , we have:

$$v \neq w \implies \mathring{B}(v, \lambda/2) \cap \mathring{B}(w, \lambda/2) = \emptyset.$$

Consequently

$$\sum_{v \in E, \|v\| \leq x} \lambda(\mathring{B}(v, \lambda/2)) = \lambda_{\overline{E}} \left(\bigcup_{v \in E, \|v\| \leq x} \mathring{B}(v, \lambda/2) \right) \leq \lambda(\mathring{B}(0, x + \lambda/2)).$$

If v_n denotes the volume of the n -dimensional unit ball, this estimate may be written:

$$v_n N_{\overline{E}}(x) (\lambda/2)^n \leq v_n (x + \lambda/2)^n.$$

This is clearly equivalent to (3.41). □

Lemma 3.6.4. *For any positive integer n and any $\beta \in]n/2, +\infty[$, we have*

$$\int_{\beta}^{+\infty} u^{n/2} e^{-u} du \leq \left(1 - \frac{n}{2\beta}\right)^{-1} \beta^{n/2} e^{-\beta}.$$

¹²since each “short vector” of \overline{E} is also a facet vector — that is, a vector v in $\mathbb{E} \setminus \{0\}$ such that the Voronoi domains of 0 and v have a non-empty intersection of dimension $n - 1$ — and, as shown by Minkowski, the number of facet vectors of a euclidean lattice of rank n is at most $2(2^n - 1)$.

Proof. The derivative $\frac{n}{2u} - 1$ of $\frac{n}{2} \log u - u$ is bounded from above by $\frac{n}{2\beta} - 1$ when u belongs to $[\beta, +\infty[$. Therefore, for any u in this interval:

$$\frac{n}{2} \log u - u \leq \left(\frac{n}{2} \log \beta - \beta\right) + (u - \beta)\left(\frac{n}{2\beta} - 1\right).$$

Consequently, we have:

$$\begin{aligned} \int_{\beta}^{+\infty} u^{n/2} e^{-u} du &= \int_{\beta}^{+\infty} \exp\left(\frac{n}{2} \log u - u\right) du \\ &\leq \exp\left(\frac{n}{2} \log \beta - \beta\right) \int_{\beta}^{+\infty} e^{-(1-n/2\beta)(u-\beta)} du = \exp\left(\frac{n}{2} \log \beta - \beta\right) (1 - n/2\beta)^{-1}. \end{aligned}$$

□

Proof of Proposition 3.6.2. From the definition of $h_{\theta}^0(\overline{E})$, Proposition 3.6.1, and the fact that

$$N_{\overline{E}}(x) = 1 \text{ if } x \in [0, \lambda[,$$

we get:

$$\begin{aligned} e^{h_{\theta}^0(\overline{E})} - 1 &= \theta_{\overline{E}}(1) - 1 = \pi \int_0^{+\infty} [N_{\overline{E}}(\sqrt{x}) - 1] e^{-\pi x} dx \\ (3.43) \quad &\leq \int_{\pi\lambda^2}^{+\infty} N_{\overline{E}}(\sqrt{u/\pi}) e^{-u} du. \end{aligned}$$

Moreover, the upper bound (3.42) on $N_{\overline{E}}$ over $[\lambda, +\infty[$ shows that

$$\int_{\pi\lambda^2}^{+\infty} N_{\overline{E}}(\sqrt{u/\pi}) e^{-u} du \leq \left(\frac{3}{\lambda\sqrt{\pi}}\right)^n \int_{\pi\lambda^2}^{+\infty} u^{n/2} e^{-u} du.$$

This establishes (3.39). Finally (3.40) follows from the integral estimate in Lemma 3.6.4, applied with $\beta = \pi\lambda^2$. □

3.7. Application to hermitian line bundles. Let \overline{L} be a hermitian line bundle over $\text{Spec } \mathcal{O}_K$.

3.7.1. The first minimum of the direct image of a hermitian line bundle. The direct image $\pi_* \overline{L}$ is an euclidean lattice of rank $n := [K : \mathbb{Q}]$ over $\text{Spec } \mathbb{Z}$. Its first minimum $\lambda_1(\pi_* \overline{L})$ admits a simple lower bound in terms of the *normalized Arakelov degree*

$$\widehat{\deg}_n \overline{L} := \frac{1}{[K : \mathbb{Q}]} \widehat{\deg} \overline{L}$$

of \overline{L} . Indeed, as shown in [Gro01], Lemma 7.1, or [BK10], Proposition 3.3.1, we have:

$$(3.44) \quad \frac{\lambda_1(\overline{E})^2}{n} \geq e^{-2\widehat{\deg}_n \overline{L}}.$$

In particular, we have:

$$(3.45) \quad \widehat{\deg}_n \overline{L} < \frac{1}{2} \log(2\pi) \implies \frac{\lambda_1(\overline{E})^2}{n} > \frac{1}{2\pi}.$$

Therefore, for any hermitian line bundle satisfying $\widehat{\deg}_n \overline{L} < (1/2) \log(2\pi)$, we may apply Proposition 3.6.2 and derive an upper bound on $h_{\theta}^0(\overline{L}) := h_{\theta}^0(\pi_* \overline{L})$.

3.7.2. *Hermitian line bundles of negative degree.* For instance, when $\widehat{\deg} \bar{L} \leq 0$, from (3.44) we obtain:

$$\lambda_1(\bar{E})^2 \geq n,$$

and the upper bounds (3.39) and (3.40) become:

$$(3.46) \quad e^{h_\theta^0(\bar{L})} - 1 \leq 3^n (1 - 1/2\pi)^{-1} e^{-\pi \lambda_1(\bar{E})^2}.$$

By using (3.44) again, we finally obtain the following minor variant of [Gro01], Proposition 7.2 (which, in a slightly less precise form, already appears as Corollary 1 to Proposition 2 in [vdGS00]):

Proposition 3.7.1. *For any hermitian line bundle \bar{L} over $\text{Spec } \mathcal{O}_K$ such that $\widehat{\deg} \bar{L} \leq 0$, we have:*

$$(3.47) \quad h_\theta^0(\bar{L}) \leq 3^{[K:\mathbb{Q}]} (1 - 1/2\pi)^{-1} \exp \left(-\pi [K:\mathbb{Q}] e^{-2\widehat{\deg}_n \bar{L}} \right).$$

The right-hand side of (3.47) is always ≤ 1 , and actually goes to zero when $[K:\mathbb{Q}]$ or $-\widehat{\deg} \bar{L}$ goes to $+\infty$. Indeed, from (3.47), we immediately obtain:

$$(3.48) \quad h_\theta^0(\bar{L}) \leq c^{[K:\mathbb{Q}]} \exp \left(-\pi [K:\mathbb{Q}] (e^{-2\widehat{\deg}_n \bar{L}} - 1) \right),$$

where

$$c := \frac{3e^{-\pi}}{1 - 1/2\pi} = 0.154180\dots$$

3.7.3. *Hermitian line bundles of positive degree.*

Proposition 3.7.2. *There exists $c \in]0, 1[$ such that, for any number field K and any hermitian line bundle \bar{L} over $\text{Spec } \mathcal{O}_K$ such that $\widehat{\deg} \bar{L} \geq 0$, we have:*

$$(3.49) \quad h_\theta^0(\bar{L}) \leq c^{[K:\mathbb{Q}]} + \widehat{\deg} \bar{L}.$$

The proof will show that we may choose the same constant c as in (3.48).

Clearly Proposition 3.7.2 establishes the validity, for any hermitian line bundle of non-negative degree over $\text{Spec } \mathcal{O}_K$, of the following inequality, familiar in a geometric context:

$$(3.50) \quad h_\theta^0(\bar{L}) \leq 1 + \widehat{\deg} \bar{L}.$$

Proposition 3.7.2 improves on [Gro01], Proposition 7.3, where a similar upper bound on $h_\theta^0(\bar{L})$ is established, with a constant of the order of $(1/2)[K:\mathbb{Q}] \log[K:\mathbb{Q}]$ instead of $c^{[K:\mathbb{Q}]}$.

Proof of Proposition 3.7.2. When $\widehat{\deg} \bar{L} = 0$, the estimate (3.49) takes the form

$$h_\theta^0(\bar{L}) \leq c^{[K:\mathbb{Q}]}$$

and follows from (3.48). To derive the general validity of (3.49) from this special case, we may use the inequality (3.24) established in Corollary 3.3.5.

Indeed, if \bar{L} is a hermitian line bundle over $\text{Spec } \mathcal{O}_K$ of non-negative Arakelov degree and if

$$\delta = \widehat{\deg}_n \bar{L} := [K:\mathbb{Q}]^{-1} \widehat{\deg} \bar{L},$$

then the hermitian line bundle $\bar{\mathcal{O}}(\delta)$ satisfies $\widehat{\deg} \bar{\mathcal{O}}(\delta) = \widehat{\deg} \bar{L}$. Therefore $\widehat{\deg} (\bar{L} \otimes \bar{\mathcal{O}}(-\delta)) = 0$ and, according to the special case above,

$$(3.51) \quad h_\theta^0(\bar{L} \otimes \bar{\mathcal{O}}(-\delta)) \leq c^{[K:\mathbb{Q}]}.$$

Besides, according to (3.25),

$$(3.52) \quad h_\theta^0(\bar{L}) \leq h_\theta^0(\bar{L} \otimes \bar{\mathcal{O}}(-\delta)) + [K:\mathbb{Q}] \delta.$$

The inequality (3.49) follows from (3.51) and (3.52). \square

3.7.4. *A scholium.* For later reference, we spell out the following straightforward consequences of the upper-bounds on $h_\theta^0(\overline{L})$ established in the previous paragraphs:

Proposition 3.7.3. *For any hermitian line bundle \overline{L} over $\text{Spec } \mathcal{O}_K$ and any $t \in \mathbb{R}$ such that $\widehat{\deg} \overline{L} \leq t$, we have :*

$$h_\theta^0(\overline{L}) \leq 1 + t \quad \text{if } t \geq 0,$$

and

$$h_\theta^0(\overline{L}) \leq \exp\left(-\pi[K : \mathbb{Q}](e^{-2t/[K:\mathbb{Q}]} - 1)\right) \leq \exp(2\pi t) \quad \text{if } t \leq 0.$$

□

3.8. Subadditivity of h_θ^0 and h_θ^1 .

3.8.1. The basic subadditivity property.

Proposition 3.8.1. *For any admissible short exact sequence of hermitian vector bundles over the arithmetic curve $\text{Spec } \mathcal{O}_K$*

$$(3.53) \quad \overline{\mathcal{E}} : 0 \longrightarrow \overline{E} \xrightarrow{i} \overline{F} \xrightarrow{p} \overline{G} \longrightarrow 0,$$

the following inequality holds:

$$(3.54) \quad h_\theta(\overline{\mathcal{E}}) := h_\theta^0(\overline{E}) - h_\theta^0(\overline{F}) + h_\theta^0(\overline{G}) \geq 0.$$

Moreover equality holds in (3.54) if and only if the admissible short exact sequence (3.53) is split.

Observe that the additivity of the Arakelov degree in short exact sequences (2.12), together with the Poisson-Riemann-Roch formula (3.14) shows that

$$(3.55) \quad h_\theta^1(\overline{E}) - h_\theta^1(\overline{F}) + h_\theta^1(\overline{G}) = h_\theta^0(\overline{E}) - h_\theta^0(\overline{F}) + h_\theta^0(\overline{G}).$$

In particular, Proposition 3.8.1 also holds with h_θ^0 replaced by h_θ^1 .

3.8.2. *Proof of Proposition 3.8.1.* When $\mathcal{O}_K = \mathbb{Z}$, Proposition 3.8.1 is established as Lemma 5.3 in [Gro01].

The inequality (3.54) actually follows from the following lemma, of independent interest, which may be seen as a “pointwise version” of (3.54) and will play a key role in deriving the main results of this article (notably to establish Lemma 8.5.1, crucial to the proof of Theorem 8.1.1 *infra*):

Lemma 3.8.2. *Consider an admissible short exact sequence of hermitian vector bundles over $\text{Spec } \mathbb{Z}$,*

$$0 \longrightarrow \overline{E} \xrightarrow{i} \overline{F} \xrightarrow{p} \overline{G} \longrightarrow 0.$$

Then, for any $g \in G$, its preimage $p^{-1}(g)$ in F satisfies:

$$(3.56) \quad \sum_{f \in p^{-1}(g)} e^{-\pi \|f\|_F^2} \leq e^{-\pi \|g\|_G^2} \sum_{e \in E} e^{-\pi \|e\|_E^2}.$$

Indeed, by summing (3.56) over $g \in G$, we get the inequality:

$$\sum_{f \in F} e^{-\pi \|f\|_F^2} \leq \sum_{g \in G} e^{-\pi \|g\|_G^2} \sum_{e \in E} e^{-\pi \|e\|_E^2}.$$

Taking the logarithms, this establishes (3.54) when $\mathcal{O}_K = \mathbb{Z}$. The case of a general number field K follows from this special case, applied to the admissible short exact sequence over $\text{Spec } \mathbb{Z}$ deduced from (3.53) by taking its direct image on $\text{Spec } \mathbb{Z}$.

Proof of Lemma 3.8.2. Let us denote by $s^\perp : G_{\mathbb{R}} \longrightarrow F_{\mathbb{R}}$ the orthogonal splitting of the surjective linear map $p_{\mathbb{R}} : F_{\mathbb{R}} \longrightarrow G_{\mathbb{R}}$. (Its image is the orthogonal complement $(\ker p_{\mathbb{R}})^\perp$ of $\ker p_{\mathbb{R}}$ in $F_{\mathbb{R}}$, defined by means of the euclidean structure on $F_{\mathbb{R}}$ attached to $\|\cdot\|_{\overline{F}}$.)

We may choose an element f_0 in $p^{-1}(g)$. Then we have:

$$p^{-1}(g) = f_0 + i(E).$$

The element $f_0 - s^\perp(g)$ of $F_{\mathbb{R}}$ belongs to $\ker p_{\mathbb{R}} = \text{im } i_{\mathbb{R}}$, and may be written $i_{\mathbb{R}}(\delta)$ for some (unique) element δ of $E_{\mathbb{R}}$. Then, for any $e \in E$, we have

$$\|f_0 + i(e)\|_{\overline{F}}^2 = \|s^\perp(g) + i(\delta + e)\|_{\overline{F}}^2 = \|g\|_{\overline{G}}^2 + \|\delta + e\|_{\overline{E}}^2.$$

Together with (3.7) (applied to $\overline{V} = \overline{E}$ and $x = \delta$), this shows that

$$\sum_{f \in p^{-1}(g)} e^{-\pi \|f\|_{\overline{F}}^2} = \sum_{e \in E} e^{-\pi (\|g\|_{\overline{G}}^2 + \|\delta + e\|_{\overline{E}}^2)} \leq e^{-\pi \|g\|_{\overline{G}}^2} \sum_{e \in E} e^{-\pi \|e\|_{\overline{E}}^2}.$$

□

To complete the proof of Proposition 3.8.1, we are left to show that equality holds in (3.54) if and only if the admissible short exact sequence (3.53) over $\text{Spec } \mathcal{O}_K$ is split.

Observe that, according to Proposition 2.4.1,

$$\pi_* \overline{\mathcal{E}} : 0 \longrightarrow \pi_* \overline{E} \xrightarrow{i} \pi_* \overline{F} \xrightarrow{p} \pi_* \overline{G} \longrightarrow 0$$

is an admissible extension of hermitian vector bundles over $\text{Spec } \mathbb{Z}$, and is split (over $\text{Spec } \mathbb{Z}$) if and only if $\overline{\mathcal{E}}$ is split (over $\text{Spec } \mathcal{O}_K$). Moreover, by the very definition of h_θ^0 , we have:

$$\begin{aligned} h_\theta(\overline{\mathcal{E}}) &= h_\theta^0(\overline{E}) - h_\theta^0(\overline{F}) + h_\theta^0(\overline{G}) \\ &= h_\theta^0(\pi_* \overline{E}) - h_\theta^0(\pi_* \overline{F}) + h_\theta^0(\pi_* \overline{G}) \\ &= h_\theta(\pi_* \overline{\mathcal{E}}). \end{aligned}$$

Therefore, to complete the proof of Proposition 3.8.1, we may assume that $\text{Spec } \mathcal{O}_K = \text{Spec } \mathbb{Z}$. In this case, it will follow from an analysis of the equality case in the above arguments.

To expound this analysis, it is convenient to introduce the following definition. Under the assumption that $\mathcal{O}_K = \mathbb{Z}$, for every T in

$$\text{Hom}_{\mathbb{C}}(G_{\mathbb{C}}, E_{\mathbb{C}})^{F^\infty} \simeq \text{Hom}_{\mathbb{Z}}(G, E) \otimes \mathbb{R} \simeq \text{Hom}_{\mathbb{R}}(G_{\mathbb{R}}, E_{\mathbb{R}}),$$

we define:

$$(3.57) \quad \text{Gext}_{\widehat{E}, \widehat{G}}(T) := \sum_{(e, g) \in E \times G} e^{-\pi (\|e - T(g)\|_{\overline{E}}^2 + \|g\|_{\overline{G}}^2)}.$$

The proof of Proposition 3.8.1 is now completed by the following Proposition, the formulation of which uses the formalism of arithmetic and admissible extensions recalled in Paragraphs 2.4.4–2.4.6:

Proposition 3.8.3. *For every $T \in \text{Hom}_{\mathbb{R}}(G_{\mathbb{R}}, E_{\mathbb{R}})$ the class of which in $\widehat{\text{Ext}}_{\mathbb{Z}}^1(G, E)$ coincides with the class of the admissible extension \mathcal{E} , the following equality holds:*

$$(3.58) \quad \exp(-h_\theta(\overline{\mathcal{E}})) = \frac{\text{Gext}_{\overline{E}, \overline{G}}(T)}{\text{Gext}_{\overline{E}, \overline{G}}(0)}.$$

Moreover we have:

$$(3.59) \quad \text{Gext}_{\overline{E}, \overline{G}}(T) = \text{covol}(\overline{E}) \sum_{(e^\vee, g) \in E^\vee \times G} e^{-\pi (\|e^\vee\|_{\overline{E}^\vee}^2 + \|g\|_{\overline{G}}^2)} e^{2\pi i \langle e^\vee \otimes g, T \rangle}.$$

and

$$(3.60) \quad 0 < \text{Gext}_{\overline{E}, \overline{G}}(T) \leq \text{Gext}_{\overline{E}, \overline{G}}(0),$$

and the equality $\text{Gext}_{\overline{E}, \overline{G}}(T) = \text{Gext}_{\overline{E}, \overline{G}}(0)$ holds if and only if T belongs to $\text{Hom}_{\mathbb{Z}}(G, E)$.

In the right side of (3.59), $e^{\vee} \otimes g$ belongs to $E^{\vee} \otimes G$ and T to

$$\text{Hom}_{\mathbb{R}}(G_{\mathbb{R}}, E_{\mathbb{R}}) \simeq G_{\mathbb{R}}^{\vee} \otimes_{\mathbb{R}} E_{\mathbb{R}} \simeq (G^{\vee} \otimes E)_{\mathbb{R}},$$

and their pairing $\langle e^{\vee} \otimes g, T \rangle$ is equal to the real number $e^{\vee}(T(g))$.

Proof. From the very definition of $\text{Gext}_{\overline{E}, \overline{G}}(T)$, we get:

$$h_{\theta}^0(\overline{E} \oplus \overline{G}^T) = \log \text{Gext}_{\overline{E}, \overline{G}}(T).$$

In particular:

$$h_{\theta}^0(\overline{E}) + h_{\theta}^0(\overline{G}) = \log \text{Gext}_{\overline{E}, \overline{G}}(0).$$

The relation (3.58) follows from these two equalities.

The equality (3.59) follows from the Poisson formula (3.4) applied to $\overline{V} = \overline{E}$ and $x = T(g)$.

The inequality $\text{Gext}_{\overline{E}, \overline{G}}(T) > 0$ is clear, and the inequality $\text{Gext}_{\overline{E}, \overline{G}}(T) \leq \text{Gext}_{\overline{E}, \overline{G}}(0)$ follows from (3.59).

Finally the expression (3.59) for $\text{Gext}_{\overline{E}, \overline{G}}(T)$ shows that, for any T in

$$\text{Hom}_{\mathbb{R}}(G_{\mathbb{R}}, E_{\mathbb{R}}) \simeq \text{Hom}_{\mathbb{Z}}(G, E) \otimes_{\mathbb{Z}} \mathbb{R},$$

the following conditions are successively equivalent:

- (1) $\text{Gext}_{\overline{E}, \overline{G}}(T) = \text{Gext}_{\overline{E}, \overline{G}}(0)$;
- (2) for any $(e^{\vee}, g) \in E^{\vee} \times G$, $e^{2\pi i \langle e^{\vee} \otimes g, T \rangle} = 1$;
- (3) for any $(e^{\vee}, g) \in E^{\vee} \times G$, $\langle e^{\vee} \otimes g, T \rangle \in \mathbb{Z}$;
- (4) T belongs to $G^{\vee} \otimes E \simeq \text{Hom}_{\mathbb{Z}}(G, E)$.

□

3.8.3. The average value of $\exp(-h_{\theta}(\overline{\mathcal{E}}))$. Proposition 3.8.3 not only makes clear the non-negativity of $h_{\theta}(\overline{\mathcal{E}})$. It also allows us to compute its “geometric average” over the arithmetic extension group $\widehat{\text{Ext}}_{\mathbb{Z}}^1(G, E)$.

The group

$$\widehat{\text{Ext}}_{\mathbb{Z}}^1(G, E) \simeq \text{Hom}_{\mathbb{Z}}(G, E) \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}$$

has indeed a natural structure of compact Lie group (it is a “compact torus” of dimension $\text{rk } E \cdot \text{rk } F$) and, as such, is equipped with a canonical Haar measure, normalized by the condition

$$\int_{\widehat{\text{Ext}}_{\mathbb{Z}}^1(G, E)} d\mu = 1.$$

Proposition 3.8.4. *Let \overline{E} and \overline{G} be two hermitian vector bundles over $\text{Spec } \mathbb{Z}$. For any $T \in \text{Hom}_{\mathbb{R}}(G_{\mathbb{R}}, E_{\mathbb{R}})$, let $[T]$ denote its class in $\widehat{\text{Ext}}_{\mathbb{Z}}^1(G, E)$ and let*

$$\overline{\mathcal{E}}(T) : 0 \longrightarrow \overline{E} \xrightarrow{i} \overline{E} \oplus \overline{G}^T \xrightarrow{p} \overline{G} \longrightarrow 0$$

be the associated admissible extension of class $[T]$ (see 2.4.6 and (2.15) supra).

Then we have:

$$(3.61) \quad \int_{[T] \in \widehat{\text{Ext}}_{\mathbb{Z}}^1(G, E)} e^{-h_{\theta}(\overline{\mathcal{E}}(T))} d\mu([T]) = 1 - \left(1 - e^{-h_{\theta}^1(\overline{E})}\right) \left(1 - e^{-h_{\theta}^0(\overline{G})}\right).$$

Proof. According to (3.58), we have:

$$(3.62) \quad \int_{[T] \in \widehat{\text{Ext}}_{\mathbb{Z}}^1(G, E)} e^{-h_{\theta}(\overline{\mathcal{E}}(T))} d\mu([T]) = \text{Gext}_{\overline{E}, \overline{G}}(0)^{-1} \int_{[T] \in \widehat{\text{Ext}}_{\mathbb{Z}}^1(G, E)} \text{Gext}_{\overline{E}, \overline{G}}(T) d\mu([T]).$$

Moreover, from (3.59), we get:

$$(3.63) \quad \text{Gext}_{\overline{E}, \overline{G}}(0) = \text{covol}(\overline{E}) \sum_{(e^\vee, g) \in E^\vee \times G} e^{-\pi(\|e^\vee\|_{\overline{E}^\vee}^2 + \|g\|_{\overline{G}}^2)}$$

$$(3.64) \quad = \text{covol}(\overline{E}) e^{h_\theta^1(\overline{E})} e^{h_\theta^0(\overline{G})},$$

and

$$(3.65) \quad \int_{[T] \in \widehat{\text{Ext}}_{\mathbb{Z}}^1(G, E)} \text{Gext}_{\overline{E}, \overline{G}}(T) d\mu([T]) = \text{covol}(\overline{E}) \sum_{\substack{(e^\vee, g) \in E^\vee \times G \\ e^\vee \otimes g = 0}} e^{-\pi(\|e^\vee\|_{\overline{E}^\vee}^2 + \|g\|_{\overline{G}}^2)}$$

$$(3.66) \quad = \text{covol}(\overline{E}) \left(\sum_{e^\vee \in E^\vee} e^{-\pi\|e^\vee\|_{\overline{E}^\vee}^2} + \sum_{g \in \times G} e^{-\pi\|g\|_{\overline{G}}^2} - 1 \right)$$

$$(3.67) \quad = \text{covol}(\overline{E}) \left(e^{h_\theta^1(\overline{E})} + e^{h_\theta^0(\overline{G})} - 1 \right).$$

Formula (3.61) follows from (3.62), (3.63) and (3.65). \square

Corollary 3.8.5. *There exists an admissible extension $\overline{\mathcal{E}}$ of \widehat{G} by \widehat{E} such that*

$$h_\theta(\overline{\mathcal{E}}) > -\log \left[1 - \left(1 - e^{-h_\theta^1(\overline{E})} \right) \left(1 - e^{-h_\theta^0(\overline{G})} \right) \right].$$

\square

4. GEOMETRY OF NUMBERS AND θ -INVARIANTS

In this section, we pursue our study of the invariants $h_\theta^i(\overline{E})$ attached to some hermitian vector bundle \overline{E} over some arithmetic curve $\text{Spec } \mathcal{O}_K$. We focus on the basic situation when $\mathcal{O}_K = \mathbb{Z}$ — that is, when \overline{E} is an euclidean lattice — and we relate the θ -invariants $h_\theta^0(\overline{E})$ and $h_\theta^1(\overline{E})$ to various invariants of \overline{E} classically considered in geometry of numbers.

The results of the present section will not be used in the next sections of this article. They are intended to clarify the meaning of the θ -invariants from the perspective of the theory of euclidean lattices, and their proofs will combine three lines of thought:

(i) The methods introduced by Banaszczyk in his work [Ban93] to establish “transference inequalities”, relating invariants of euclidean lattices and of their dual lattices, with essentially optimal constants¹³.

(ii) An extended version of Cramér’s theory of large deviations, valid on some measure space of infinite total mass. This theory is presented in Appendix A, and the reader will find a self-contained summary of its results required for our study of euclidean lattices in section A.5, where they are stated in a form that emphasizes their relations with the formalism of statistical physics.

(iii) Siegel’s mean value theorem and its use for showing the existence of euclidean lattices with remarkable density properties (*à la* Minkowski-Hlawka) by probabilistic arguments.

As explained in the introduction, Banaszczyk’s techniques allows us to establish notably that the invariants $h_\theta^0(\overline{E})$ and $h_{\text{Ar}}^0(\overline{E})$ of some euclidean lattices \overline{E} differ by some error term bounded in terms of the rank of E only (see (1.6) and Theorem 4.1.1). Although these techniques have been developed

¹³See [Gas99], Section 3, for an early application of these methods to Arakelov geometry. Banaszczyk results have played an important role in lattice-based cryptography, and we refer the reader to [MR07] and [TLX14] for improvements and applications of the original results in [Ban93] related to lattice-based cryptography, and for further references.

independently, they constitute a tool of choice in the study of the θ -invariants. Conversely, the use of θ -invariants shed some light on Banaszczyk's arguments and, in Section 4.2, we give a self-contained presentation of some of the most important results in [Ban93] from this perspective.

The large deviation theorems in Appendix A admit as consequences the duality relations (1.7) and (1.8) between the asymptotic invariant $\tilde{h}_{\text{Ar}}^0(\overline{E}, t)$ and the function $\log \theta_{\overline{E}}$. Moreover their thermodynamical interpretation leads us to the expression

$$(4.1) \quad \tilde{h}_{\text{Ar}}^0(\overline{E}_1 \oplus \overline{E}_2, t) = \max_{\substack{t_1, t_2 > 0 \\ t_1 + t_2 = t}} \left(\tilde{h}_{\text{Ar}}^0(\overline{E}_1, t_1) + \tilde{h}_{\text{Ar}}^0(\overline{E}_2, t_2) \right).$$

for the asymptotic invariant $\tilde{h}_{\text{Ar}}^0(\overline{E}_1 \oplus \overline{E}_2, t)$ attached to the direct sum $\overline{E}_1 \oplus \overline{E}_2$ of two euclidean lattices \overline{E}_1 and \overline{E}_2 . Formula (4.1) may actually be understood as an avatar of the second law of thermodynamics (see Proposition 4.4.6).

The probabilistic arguments based on Siegel's theorem mean value theorem will notably establish the existence, for any (n, δ) in $\mathbb{N}_{\geq 2} \times \mathbb{R}$, of some euclidean lattice \overline{E} of rank n and degree δ such that

$$h_{\theta}^0(\overline{E}) < \log(1 + e^{\delta}).$$

These probabilistic arguments will also show that the constants in diverse comparison estimates established in this section are essentially optimal.

4.1. Comparing h_{θ}^0 and h_{Ar}^0 .

4.1.1. *The invariants $h_{\text{Ar}}^0(\overline{E})$, $h_{\text{Ar}-}^0(\overline{E})$, and $h_{\text{Bl}}^0(\overline{E})$.* Following [GS91], we define:

$$h_{\text{Ar}}^0(\overline{E}) := \log |\{v \in E \mid \|v\| \leq 1\}|.$$

This definition is motivated by the classical philosophy in Arakelov geometry (see for instance [Man85], Section 2.3) according to which the finite set

$$\{v \in E \mid \|v\| \leq 1\}$$

should be interpreted as the “global sections” of the hermitian vector bundle \overline{E} over $\text{Spec } \mathbb{Z}$ “completed” by its archimedean place.

It turns out that the invariant $h_{\text{Ar}}^0(\overline{E})$, defined as above as the logarithm of the number of points in the unit ball of the euclidean lattice \overline{E} , and the θ -invariant $h_{\theta}^0(\overline{E})$ coincide up to an error term bounded in function of $\text{rk } E$ only:

Theorem 4.1.1. *For any euclidean lattice of positive rank \overline{E} , we have:*

$$(4.2) \quad h_{\theta}^0(\overline{E}) - \frac{1}{2} \text{rk } E \cdot \log \text{rk } E + \log(1 - 1/2\pi) \leq h_{\text{Ar}}^0(\overline{E}) \leq h_{\theta}^0(\overline{E}) + \pi.$$

We shall actually establish a slightly stronger version of the first inequality in (4.2), namely¹⁴:

$$(4.3) \quad h_{\text{Ar}-}^0(\overline{E}) := \log |\{v \in E \mid \|v\| < 1\}| \geq h_{\theta}^0(\overline{E}) + \log(1 - 1/2\pi) - \frac{1}{2} \text{rk } E \cdot \log \text{rk } E.$$

Together with the “Riemann inequality” over $\text{Spec } \mathbb{Z}$ (3.15), inequality (4.3) entails the following strengthened variant of Minkowski's First Theorem:

$$h_{\text{Ar}-}^0(\overline{E}) \geq \log(1 - 1/2\pi) - \frac{1}{2} \text{rk } E \cdot \log \text{rk } E + \widehat{\deg } \overline{E}.$$

Corollary 4.1.2. *For any euclidean lattice of positive rank \overline{E} ,*

$$\lambda_1(\overline{E}) \geq 1 \implies h_{\theta}^0(\overline{E}) \leq -\log(1 - 1/2\pi) + \frac{1}{2} \text{rk } E \cdot \log \text{rk } E.$$

¹⁴Since $h_{\text{Ar}-}^0(\overline{E}) = h_{\text{Ar}}^0(\overline{E} \otimes \overline{\mathcal{O}}(-\epsilon))$ for any small enough ϵ in \mathbb{R}_+^* , (4.3) actually follows from (4.2) applied to $\overline{E} \otimes \overline{\mathcal{O}}(-\epsilon)$, by taking the limit $\epsilon \rightarrow 0_+$.

Proof. The condition $\lambda_1(\overline{E}) \geq 1$ is equivalent to the equality $h_{\text{Ar}-}^0(\overline{E}) = 0$. \square

Observe that, from Proposition 3.6.2, we also obtain a lower bound on $h_\theta^0(\overline{E})$ for any euclidean lattice \overline{E} with $n := \text{rk } E > 0$ and $\lambda_1(\overline{E}) \geq 1$, namely:

$$h_\theta^0(\overline{E}) \leq \log(1 + C(n, 1)) = \log \left(1 + (3/\sqrt{\pi})^n \int_{\pi}^{+\infty} u^{n/2} e^{-u} du \right).$$

This estimate is slightly weaker than the one in Corollary 4.1.2. Indeed, when n goes to $+\infty$, its right hand side is of the form $\frac{1}{2}n \cdot \log n + \alpha \cdot n + o(n)$ with $\alpha > 0$.

We may also introduce a variant *à la Blichfeldt* of $h_{\text{Ar}}^0(\overline{E})$, namely

$$h_{\text{Bl}}^0(\overline{E}) := \log \max_{x \in E_{\mathbb{R}}} |\{v \in E \mid \|v - x\| \leq 1\}|$$

(see Section 4.3.2 *infra*). It is straightforward that

$$h_{\text{Ar}-}^0(\overline{E}) \leq h_{\text{Ar}}^0(\overline{E}) \leq h_{\text{Bl}}^0(\overline{E}),$$

and we shall establish the following strengthening of the second inequality in (4.2):

Proposition 4.1.3. *For any euclidean lattice of positive rank \overline{E} , we have:*

$$(4.4) \quad h_{\text{Bl}}^0(\overline{E}) \leq h_\theta^0(\overline{E}) + \pi.$$

Here is an elementary consequence of Theorem 4.1.1 and Proposition 4.1.3 which does not seem to appear in the literature:

Corollary 4.1.4. *For any euclidean lattice of positive rank \overline{E} ,*

$$h_{\text{Bl}}^0(\overline{E}) \leq \pi - \log(1 - 1/2\pi) + \frac{1}{2} \text{rk } E \cdot \log \text{rk } E + h_{\text{Ar}-}^0(\overline{E}).$$

\square

4.1.2. *Proof of Proposition 4.1.3 and Theorem 4.1.1.* To prove Proposition 4.1.3, namely that

$$h_{\text{Bl}}^0(\overline{E}) \leq h_\theta^0(\overline{E}) + \pi,$$

we simply observe that, from the very definition of $h_\theta^0(\overline{E})$ and the inequality (3.7), we have, for any $x \in E_{\mathbb{R}}$,

$$h_\theta^0(\overline{E}) = \log \sum_{v \in E} e^{-\pi \|v\|^2} \geq \log \sum_{v \in E} e^{-\pi \|v-x\|^2} \geq \log \sum_{v \in E, \|v-x\| \leq 1} e^{-\pi \|v-x\|^2}$$

and that the last sum is at least

$$e^{-\pi} \cdot |\{v \in E \mid \|v - x\| \leq 1\}|.$$

The second inequality in (4.2), namely $h_{\text{Ar}}^0(\overline{E}) \leq h_\theta^0(\overline{E}) + \pi$, follows from the special case $x = 0$ (which does not request the use of (3.7)) of the previous argument.

We split the proof of (4.3) in a succession of auxiliary statements, of independent interest. The following assertions are variants of results in [Ban93], Section 1.

Lemma 4.1.5. *1) The expression $\log \theta_{\overline{E}}(t)$ defines a decreasing function of t in \mathbb{R}_+^* , and the expression*

$$(4.5) \quad \log \theta_{\overline{E}}(t) + \frac{1}{2} \text{rk } E \cdot \log t$$

an increasing function of t in \mathbb{R}_+^ .*

2) We have:

$$(4.6) \quad \sum_{v \in E} \|v\|^2 e^{-\pi t \|v\|^2} \leq \frac{\text{rk } E}{2\pi t} \sum_{v \in E} e^{-\pi t \|v\|^2}.$$

3) For any t and r in \mathbb{R}_+^* , we have:

$$(4.7) \quad \sum_{v \in E, \|v\| < r} e^{-\pi t \|v\|^2} \geq \left(1 - \frac{\text{rk } E}{2\pi t r^2}\right) \sum_{v \in E} e^{-\pi t \|v\|^2}.$$

Proof. The first assertion in 1) is clear. According to the Functional Equation (3.37), the expression (4.5) may also be written

$$\widehat{\deg \overline{E}} + \log \theta_{\overline{E}}(t^{-1}),$$

and consequently defines an increasing function of t . (This fact is also a reformulation of Lemma 3.3.5 in the special case $K = \mathbb{Q}$.) The inequality (4.6) may also be written

$$-\frac{1}{\pi} \frac{d\theta_{\overline{E}}(t)}{dt} \leq \frac{\text{rk } E}{2\pi t} \theta_{\overline{E}}(t),$$

and simply expresses that the derivative of (4.5) is non-negative.

To establish the inequality (4.7), we combine the straightforward estimate

$$\sum_{v \in E, \|v\| \geq r} e^{-\pi t \|v\|^2} \leq \frac{1}{r^2} \sum_{v \in E} \|v\|^2 e^{-\pi t \|v\|^2}$$

with (4.6). This yields:

$$\sum_{v \in E, \|v\| \geq r} e^{-\pi t \|v\|^2} \leq \frac{\text{rk } E}{2\pi t r^2} \sum_{v \in E} e^{-\pi t \|v\|^2},$$

or equivalently,

$$\sum_{v \in E, \|v\| < r} e^{-\pi t \|v\|^2} \geq \left(1 - \frac{\text{rk } E}{2\pi t r^2}\right) \sum_{v \in E} e^{-\pi t \|v\|^2}.$$

□

From (4.7) with $r = 1$, we obtain that, for any $t > \text{rk } E / 2\pi$, we have:

$$h_{\text{Ar}-}^0(\overline{E}) \geq \log(1 - \text{rk } E / (2\pi t)) + \log \theta_{\overline{E}}(t).$$

Using also that, for any $t \geq 1$,

$$\log \theta_{\overline{E}}(t) \geq \log \theta_{\overline{E}}(1) - \frac{1}{2} \text{rk } E \cdot \log t,$$

we finally obtain:

Proposition 4.1.6. *For any $t \geq \min(1, \text{rk } E / 2\pi)$, we have*

$$h_{\text{Ar}-}^0(\overline{E}) \geq \log(1 - \text{rk } E / (2\pi t)) - \frac{1}{2} \text{rk } E \cdot \log t + h_{\theta}^0(\overline{E}).$$

Notably we may choose $t = \text{rk } E$, and then we obtain¹⁵:

$$h_{\text{Ar}-}^0(\overline{E}) \geq \log(1 - 1/2\pi) - \frac{1}{2} \text{rk } E \cdot \log \text{rk } E + h_{\theta}^0(\overline{E}).$$

This establishes the inequality (4.3).

4.2. Banaszczyk's estimates and θ -invariants. In the next paragraphs, we want to discuss in more details the relation between Banaszczyk's celebrated results in geometry of numbers ([Ban93]; see also [MR07] and [TLX14] for some recent developments) and the properties of θ -invariants.

This relation already appeared in the derivation of the estimates relating the invariants $h_{\theta}^0(\overline{E})$, $h_{\text{Ar}}^0(\overline{E})$ and $h_{\text{Bl}}^0(\overline{E})$ in the paragraph 4.1.2.

¹⁵The "optimal" choice of t in terms of $n := \text{rk } E$ would be $t = (n+2)/2\pi$. This choice leads to the slightly stronger estimate: $h_{\text{Ar}-}^0(\overline{E}) \geq -\frac{n+2}{2} \log \frac{n+2}{2\pi} - \log \pi + h_{\theta}^0(\overline{E})$.

4.2.1. *Banaszczyk's key estimate.* The starting point of Banaszczyk results is arguably the following estimate, which actually is a simple consequence of the increasing character of the function $\log \theta_{\overline{E}}(t) + \frac{1}{2} \text{rk } E \cdot \log t$ established in Lemma 3.3.5 and Lemma 4.1.5, 2):

Lemma 4.2.1. *Let \overline{E} be an euclidean lattice of positive rank n , and let x an element of $E_{\mathbb{R}}$.*

For any $r \in \mathbb{R}^+$ and any $t \in]0, 1]$, we have:

$$(4.8) \quad \sum_{v \in E, \|v-x\| \geq r} e^{-\pi \|v-x\|^2} \leq t^{-n/2} e^{-\pi(1-t)r^2} \sum_{v \in E} e^{-\pi \|v\|^2}.$$

Proof. We have:

$$(4.9) \quad \begin{aligned} \sum_{v \in E, \|v-x\| \geq r} e^{-\pi \|v-x\|^2} &= \sum_{v \in E, \|v-x\| \geq r} e^{-\pi(1-t)\|v-x\|^2} e^{-\pi t\|v-x\|^2} \\ &\leq e^{-\pi(1-t)r^2} \sum_{v \in E, \|v-x\| \geq r} e^{-\pi t\|v-x\|^2} \\ &\leq e^{-\pi(1-t)r^2} \sum_{v \in E, \|v-x\| \geq r} e^{-\pi t\|v\|^2} \\ (4.10) \quad &\leq e^{-\pi(1-t)r^2} t^{-n/2} \sum_{v \in E} e^{-\pi \|v\|^2}. \end{aligned}$$

Indeed, the inequality (4.9) follows from (3.7), and (4.10) from the inequality

$$\log \theta_{\overline{E}}(t) + \frac{n}{2} \log t \leq \log \theta_{\overline{E}}(1).$$

□

Clearly the inequality (4.8) is significant only for values of r such that

$$\inf_{t \in]0, 1]} t^{-n/2} e^{-\pi(1-t)r^2} \leq 1.$$

An elementary computation shows that this holds precisely when $r \geq \sqrt{\frac{n}{2\pi}}$, and that, when this holds, if we write

$$r = \sqrt{\frac{n}{2\pi}} \tilde{r}$$

with \tilde{r} in $[1, +\infty[$, then the minimum of $t^{-n/2} e^{-\pi(1-t)r^2}$ over $]0, 1]$ is achieved for $t = t_{\min} := \tilde{r}^{-2}$, and takes the value

$$t_{\min}^{-n/2} e^{-\pi(1-t_{\min})r^2} = [\tilde{r} e^{-(1/2)(\tilde{r}^2-1)}]^n.$$

It is straightforward that this minimum $[\tilde{r} e^{-(1/2)(\tilde{r}^2-1)}]^n$ is a decreasing function of $\tilde{r} \in [1, +\infty[$, which takes the value 1 when $\tilde{r} = 1$.

These elementary considerations show that Lemma 4.2.1 may be reformulated in the following version, better suited to applications:

Proposition 4.2.2. *Let \overline{E} be an euclidean lattice of positive rank n , and let x an element of $E_{\mathbb{R}}$. For every element \tilde{r} in $[1, +\infty[$, if we let*

$$r = \sqrt{\frac{n}{2\pi}} \tilde{r},$$

then we have:

$$(4.11) \quad \sum_{v \in E, \|v-x\| \geq r} e^{-\pi \|v-x\|^2} \leq [\tilde{r} e^{-(1/2)(\tilde{r}^2-1)}]^n \sum_{v \in E} e^{-\pi \|v\|^2}.$$

□

4.2.2. *Application: first minimum and θ -invariants.* The special case of (4.11) where $x = 0$ has already the following non-trivial consequence:

Corollary 4.2.3. *Let \overline{E} be an euclidean lattice of positive rank n , and of first minimum $\lambda_1(\overline{E}) \geq \sqrt{n/2\pi}$.*

If we define $\tilde{\lambda}$ in $[1, +\infty[$ by the relation:

$$\lambda_1(\overline{E}) = \sqrt{\frac{n}{2\pi}} \tilde{\lambda},$$

then the following inequality holds:

$$(4.12) \quad h_\theta^0(\overline{E}) \leq \log(1 - [\tilde{\lambda}e^{-(1/2)(\tilde{\lambda}^2-1)}]^n)^{-1}.$$

and

$$(4.13) \quad e^{h_\theta^0(\overline{E})} - 1 \leq \frac{[\tilde{\lambda}e^{-(1/2)(\tilde{\lambda}^2-1)}]^n}{1 - [\tilde{\lambda}e^{-(1/2)(\tilde{\lambda}^2-1)}]^n}.$$

Proof. By applying (4.11) to $x = 0$ and $r = \lambda_1(\overline{E})$, we get:

$$\sum_{v \in E} e^{-\pi\|v\|^2} - 1 = \sum_{v \in E, \|v\| \geq \lambda_1(\overline{E})} e^{-\pi\|v\|^2} \leq [\tilde{\lambda}e^{-(1/2)(\tilde{\lambda}^2-1)}]^n \sum_{v \in E} e^{-\pi\|v\|^2}.$$

This establishes the inequality

$$e^{h_\theta^0(\overline{E})} - 1 \leq [\tilde{\lambda}e^{-(1/2)(\tilde{\lambda}^2-1)}]^n e^{h_\theta^0(\overline{E})},$$

which in turn is equivalent to (4.12) and (4.13). \square

We may compare the upper-bound (4.13) on $h_\theta^0(\overline{E})$ in terms of the first minimum $\lambda_1(\overline{E})$, assumed to be $> (n/2\pi)^{1/2}$, obtained by means of Banaszczyk's methods, with the upper-bound derived in Proposition 3.6.2 by using Groenewegen's argument, namely:

$$(4.14) \quad e^{h_\theta^0(\overline{E})} - 1 \leq 3^n \left(1 - \frac{n}{2\pi\lambda_1(\overline{E})^2}\right)^{-1} e^{-\pi\lambda_1(\overline{E})^2}.$$

To achieve this comparison, observe that

$$(4.15) \quad [\tilde{\lambda}e^{-(1/2)(\tilde{\lambda}^2-1)}]^n = \left(\frac{n}{2\pi e}\right)^{-n/2} \lambda_1(\overline{E})^n e^{-\pi\lambda_1(\overline{E})^2}.$$

This shows that, when $\lambda_1(\overline{E})$ goes to $+\infty$, Groenewegen's bound (4.14) is better than (4.13) by a factor

$$3^n \left(\frac{n}{2\pi e}\right)^{n/2} \lambda_1(\overline{E})^{-n} = \left(\frac{3}{\sqrt{e}}\right)^n \tilde{\lambda}^{-n}.$$

Besides, for any fixed value of n , Groenewegen's bound is also better than Banaszczyk's when $\lambda_1(\overline{E})$ goes to $(n/2\pi)^{1/2}$. However, when $\tilde{\lambda} = \sqrt{\pi}$, Banaszczyk's bound improves on Groenewegen's one by a factor

$$(1 - \pi^{-1})^{-1} \left(\frac{\pi e}{3}\right)^{n/2}$$

when n goes to infinity. (Observe that $\pi e/3 = 0.974... < 1$.)

4.2.3. *Covering radius and Banaszczyk's transference estimate.* At this stage, we can easily recover Banaszczyk's transference estimate relating the first minimum and the covering radius of some euclidean lattice and of its dual.

Recall that the *covering radius* of an euclidean lattice \overline{E} of positive rank is defined as the positive real number

$$\rho(\overline{E}) := \max_{x \in E_{\mathbb{R}}} \min_{v \in E} \|v - x\| = \inf \left\{ r \in \mathbb{R}^+ \mid \bigcup_{v \in E} \overset{\circ}{B}(v, r) = E_{\mathbb{R}} \right\}$$

and that Banaszczyk's transference estimate is the second estimate in the following proposition:

Proposition 4.2.4 ([Ban93], Theorem 2.2). *For any euclidean lattice \overline{E} of positive rank n , we have:*

$$1/2 \leq \rho(\overline{E}) \lambda_1(\overline{E}^{\vee}) \leq n/2.$$

The first inequality $1/2 \leq \rho(\overline{E}) \lambda_1(\overline{E}^{\vee})$ is elementary¹⁶. To establish the second one, we first derive another corollary of Proposition 4.2.2:

Corollary 4.2.5. *Let \overline{E} be an euclidean lattice of positive rank n , and of covering radius $\rho(\overline{E}) \geq \sqrt{n/2\pi}$.*

If we define $\tilde{\rho}$ in $[1, +\infty[$ by the relation

$$\rho(\overline{E}) = \sqrt{\frac{n}{2\pi}} \tilde{\rho},$$

then there exists x in $E_{\mathbb{R}}$ such that

$$(4.16) \quad \frac{\sum_{v \in E} e^{-\pi \|v-x\|^2}}{\sum_{v \in E} e^{-\pi \|v\|^2}} \leq [\tilde{\rho} e^{-(\tilde{\rho}^2-1)/2}]^n.$$

Proof. By the very definition of $\rho(\overline{E})$, there exists x in $E_{\mathbb{R}}$ such that $\|v - x\| \geq \rho(\overline{E})$ for every v in E . For this choice of x , (4.16) follows from (4.11) applied with $r = \rho(\overline{E})$. \square

The following lemma is a straightforward reformulation of already established properties of h_{θ}^0 .

Lemma 4.2.6. *Let \overline{E} be an euclidean lattice of positive rank n .*

1) For every x in $E_{\mathbb{R}}$, we have:

$$(4.17) \quad \frac{\sum_{v \in E} e^{-\pi \|v-x\|^2}}{\sum_{v \in E} e^{-\pi \|v\|^2}} \geq 2e^{-h_{\theta}^0(\overline{E}^{\vee})} - 1.$$

2) Assume that the first minimum of the dual lattice \overline{E}^{\vee} satisfies $\lambda_1(\overline{E}^{\vee}) \geq \sqrt{n/2\pi}$. If we define $\tilde{\lambda}^{\vee}$ in $[1, +\infty[$ by the relation:

$$\lambda_1(\overline{E}^{\vee}) = \sqrt{\frac{n}{2\pi}} \tilde{\lambda}^{\vee},$$

then we have:

$$(4.18) \quad 2e^{-h_{\theta}^0(\overline{E}^{\vee})} - 1 \geq 1 - 2[\tilde{\lambda}^{\vee} e^{-(\tilde{\lambda}^{\vee 2}-1)/2}]^n.$$

Proof. 1) According to (3.8),

$$\frac{\sum_{v \in E} e^{-\pi \|v-x\|^2}}{\sum_{v \in E} e^{-\pi \|v\|^2}} \geq \frac{2}{(\text{covol } \overline{E}) \sum_{v \in E} e^{-\pi \|v\|^2}} - 1.$$

We conclude the proof of (4.17) by using the Poisson formula (3.6) and the definition of $h_{\theta}^0(\overline{E}^{\vee})$.

¹⁶Simply consider a "short vector" ξ of \overline{E}^{\vee} , an element $v \in E_{\mathbb{R}}$ such that $\xi(x) = 1/2$ and $v \in E$ such that $\|v - x\| \leq \rho(\overline{E})$, and observe that $\rho(\overline{E}) \lambda_1(\overline{E}^{\vee}) \geq \|\xi\| \cdot \|v - x\| \geq |\xi(v) - \xi(x)| \geq 1/2$.

2) To prove (4.18), we just apply the bound (4.12) to the dual lattice \overline{E}^\vee . \square

From Corollary 4.2.5 and Lemma 4.2.6, we easily derive Banaszczyk's transference estimate in the following more precise form:

Proposition 4.2.7. *Let $\psi : [1, +\infty[\xrightarrow{\sim}]0, 1]$ be the decreasing homeomorphism defined by $\psi(t) := te^{-(t^2-1)/2}$, and for every positive integer n , let $t_n := \psi^{-1}(3^{-1/n})$.*

Then we have:

$$(4.19) \quad t_n = 1 + \sqrt{\log 3/n} + O(1/n) \text{ when } n \rightarrow +\infty.$$

and, for every positive integer n :

$$(4.20) \quad t_n \leq 1 + \sqrt{\log 3/n}.$$

Moreover, for any euclidean lattice \overline{E} of positive rank n , we have:

$$(4.21) \quad \rho(\overline{E}) \lambda_1(\overline{E}^\vee) \leq \frac{t_n^2 n}{2\pi}.$$

Observe that, according to (4.20), for any $n \geq 3$, we have:

$$t_n \leq 1 + \sqrt{(\log 3)/3} = 1.605... \leq \sqrt{\pi} = 1.772...$$

Consequently Proposition 4.2.7 implies Banaszczyk's inequality $\rho(\overline{E}) \lambda_1(\overline{E}^\vee) \leq n/2$ when $n \geq 3$. This inequality is trivial when $n = 1$. When $n = 2$, it follows from elementary considerations involving reduced bases¹⁷ of \overline{E} and \overline{E}^\vee .

Proof of Proposition 4.2.7. We leave the derivation of (4.19) and (4.20) as elementary exercises.

Let consider a euclidean lattice \overline{E} of positive rank n and let us define $\tilde{\rho}$ and $\tilde{\lambda}^\vee$ as in Corollary 4.2.5 and Lemma 4.2.6. From the estimates (4.16), (4.17) and (4.18), it follows that

$$\text{if } \tilde{\rho} \text{ and } \tilde{\lambda}^\vee \text{ are } \geq 1, \text{ then } \psi(\tilde{\rho})^n + 2\psi(\tilde{\lambda}^\vee)^n \leq 1.$$

Consequently, if for some $t \in \mathbb{R}_+^*$, we have $\tilde{\rho} = \tilde{\lambda}^\vee = t$, then $3\psi(t)^n \leq 1$ if $t \geq 1$, and therefore $t \leq t_n$ and

$$\rho(\overline{E}) = \lambda_1(\overline{E}^\vee) \leq t_n \sqrt{n/2\pi}.$$

This establishes (4.21) when $\rho(\overline{E}) = \lambda_1(\overline{E}^\vee)$.

To derive the general validity of (4.21) from this special case, simply observe that replacing the euclidean lattice \overline{E} by $\overline{E} \otimes \overline{\mathcal{O}}(\delta)$ for some $\delta \in \mathbb{R}$ (that is, scaling the metric of \overline{E} by a positive factor $e^{-\delta}$) does not change the product $\rho(\overline{E}) \lambda_1(\overline{E}^\vee)$ and that, by a suitable choice of δ , the condition

$$\rho(\overline{E} \otimes \overline{\mathcal{O}}(\delta)) = \lambda_1((\overline{E} \otimes \overline{\mathcal{O}}(\delta))^\vee)$$

may be achieved. Indeed, from the very definitions of the covering radius and of the first minimum, we obtain:

$$\rho(\overline{E} \otimes \overline{\mathcal{O}}(\delta)) = e^{-\delta} \rho(\overline{E})$$

and

$$\lambda_1((\overline{E} \otimes \overline{\mathcal{O}}(\delta))^\vee) = e^\delta \lambda_1(\overline{E}^\vee).$$

\square

4.3. Subadditive invariants of euclidean lattices.

¹⁷Indeed to establish Banaszczyk's inequality for euclidean lattices of rank 2, it is enough to prove it for the euclidean lattices \overline{E}_τ , defined as $\mathbb{Z} + \tau\mathbb{Z}$ equipped with the usual complex absolute value $|\cdot|$, when τ is an element of the upper half-plane in the usual "reduction domain" defined by $\tau \geq 1$ and $|\operatorname{Re} \tau| \leq 1/2$. For such lattices, Banaszczyk's inequality takes the form: $\rho(\overline{E}_\tau^\vee) \leq \operatorname{Im} \tau$. This last estimate follows from the observation that, for any $z \in \mathbb{C}$ such that $|\operatorname{Im} z| \leq \operatorname{Im} \tau/2$, if we denote the integer closest to $\operatorname{Re} z$ by k , we have: $|z - k| \leq \sqrt{1 + (\operatorname{Im} \tau)^2/2} \leq \operatorname{Im} \tau$.

4.3.1. *Alternating inequalities.* With the notation of Proposition 3.8.1, one has the following series of alternating inequalities:

$$(4.22) \quad h_\theta^0(\overline{E}) \geq 0,$$

$$(4.23) \quad h_\theta^0(\overline{E}) - h_\theta^0(\overline{F}) \leq 0,$$

$$(4.24) \quad h_\theta^0(\overline{E}) - h_\theta^0(\overline{F}) + h_\theta^0(\overline{G}) \geq 0,$$

$$(4.25) \quad h_\theta^0(\overline{E}) - h_\theta^0(\overline{F}) + h_\theta^0(\overline{G}) - h_\theta^1(\overline{E}) \leq 0,$$

$$(4.26) \quad h_\theta^0(\overline{E}) - h_\theta^0(\overline{F}) + h_\theta^0(\overline{G}) - h_\theta^1(\overline{E}) + h_\theta^1(\overline{F}) \geq 0,$$

$$(4.27) \quad h_\theta^0(\overline{E}) - h_\theta^0(\overline{F}) + h_\theta^0(\overline{G}) - h_\theta^1(\overline{E}) + h_\theta^1(\overline{F}) - h_\theta^1(\overline{G}) = 0.$$

These are precisely the inequalities we should obtain if the h_θ^k 's were some dimensions of the spaces in a long exact cohomology sequence derived from the admissible short exact sequence

$$0 \longrightarrow \overline{E} \xrightarrow{i} \overline{F} \xrightarrow{p} \overline{G} \longrightarrow 0,$$

which would vanish in cohomological degree $k > 1$.

Indeed, the inequalities (4.22)-(4.24) have already been established. As already observed (see (3.55) *supra*), thanks to the Poisson-Riemann-Roch formula (3.11), the equality (4.27) may be written

$$\widehat{\deg} \overline{E} - \widehat{\deg} \overline{F} + \widehat{\deg} \overline{G} = 0$$

and precisely expresses the additivity (2.12) of the Arakelov degree in admissible short exact sequences. Taking (4.27) into account, inequalities (4.25) and (4.26) are equivalent to the relations $-h_\theta^1(\overline{F}) + h_\theta^1(\overline{G}) \leq 0$ and $h_\theta^1(\overline{G}) \geq 0$, which follow from Proposition 3.3.2 applied to $\phi = p$ and from Lemma 3.3.1.

Observe that, using again the Poisson-Riemann-Roch formula (3.14), the inequality (4.25) may be also written as

$$(4.28) \quad h_\theta^0(\overline{G}) \leq h_\theta^0(\overline{F}) - \widehat{\deg} \overline{E} + \frac{1}{2} \log |\Delta_K| \cdot \text{rk } E$$

4.3.2. *Blichfeldt pairs.* Let us indicate that the occurrence in geometry of numbers of “alternating inequalities”, similar to the ones satisfied by dimensions of cohomology groups, has been observed by Gillet, Mazur, and Soulé ([GMS91]) in the context of the classical theorem of Blichfeldt.

In *loc. cit.*, instead of euclidean lattices $\overline{E} := (E, \|\cdot\|)$, defined by finitely generated free \mathbb{Z} -module E and a euclidean norm $\|\cdot\|$ on $E_{\mathbb{R}}$, the authors deal with so-called *Blichfeldt pairs*

$$\mathcal{E} := (E, \mathbf{B}),$$

defined by a \mathbb{Z} -module E as above and a bounded Lebesgue measurable subset \mathbf{B} in $E_{\mathbb{R}}$. They define

$$h^0(\mathcal{E}) := \log \max_{v \in E_{\mathbb{R}}} |E \cap (v + \mathbf{B})|$$

and

$$h^1(\mathcal{E}) := h^0(\mathcal{E}) - \log \mu(\mathbf{B})$$

where μ is the Haar measure on $E_{\mathbb{R}}$ which gives a fundamental domain for E measure equal to one. They define an *exact sequence of Blichfeldt pairs*

$$0 \longrightarrow (E, \mathbf{B}) \xrightarrow{i} (F, \mathbf{C}) \xrightarrow{p} (G, \mathbf{D}) \longrightarrow 0$$

as an exact sequence of \mathbb{Z} -modules

$$0 \longrightarrow E \xrightarrow{i} F \xrightarrow{p} G \longrightarrow 0$$

such that $p_{\mathbb{R}}(\mathbf{C})$ and \mathbf{D} (resp., for any $x \in \mathbf{C}$, $p_{\mathbb{R}}^{-1}(p_{\mathbb{R}}(x)) \cap \mathbf{C}$ and $i_{\mathbb{R}}(\mathbf{B})$) coincide up to translation in $G_{\mathbb{R}}$ (resp., in $F_{\mathbb{R}}$). Then they establish the validity of (4.22)-(4.27), with h^k instead of h_θ^k , for any short exact sequence of Blichfeldt pairs as above.

It is also possible to define the direct sum of two Blichfeldt pairs $\mathcal{E}_1 := (E_1, \mathbf{B}_1)$ and $\mathcal{E}_2 := (E_2, \mathbf{B}_2)$ as

$$\mathcal{E}_1 \oplus \mathcal{E}_2 := (E_1 \oplus E_2, \mathbf{B}_1 \times \mathbf{B}_2).$$

Then the following additivity relations are easily established:

$$h^k(\mathcal{E}_1 \oplus \mathcal{E}_2) = h^k(\mathcal{E}_1) + h^k(\mathcal{E}_2), \text{ for } k = 0, 1.$$

To any euclidean lattice $\overline{E} = (E, \|\cdot\|)$, we may attach a natural Blichfeldt pair, namely

$$\mathcal{E} := (E, \mathbf{B}(\overline{E}_{\mathbb{R}})),$$

where

$$\mathbf{B}(\overline{E}_{\mathbb{R}}) := \{v \in E_{\mathbb{R}} \mid \|v\| \leq 1\}.$$

Observe that, with the notation of Section 4.1, we have:

$$h^0(\mathcal{E}) = h_{\text{Bl}}^0(\overline{E}).$$

However this construction of Blichfeldt pairs from euclidean lattices is not compatible with short exact sequences or products of euclidean lattices and Blichfeldt pairs. Actually, for any two euclidean lattices \overline{E}_1 and \overline{E}_2 of positive ranks, the Blichfeldt pair associated to their direct sum $\overline{E}_1 \oplus \overline{E}_2$ cannot be expressed as the direct sum of two Blichfeldt pairs.

This lack of compatibility prevents one to associate invariants h^0 and h^1 to euclidean lattices, so that they would satisfy the alternating inequalities (4.22)-(4.27), by reducing to the construction in [GMS91].

4.3.3. Concerning the subadditivity of h_{Ar}^0 . At this point, it may be worth to emphasize that the subadditivity property (3.54) (or equivalently (4.25)) satisfied by h_{θ}^0 does *not* hold when h_{θ}^0 is replaced by h_{Ar}^0 (contrary to what is claimed in [GS91], p. 356, Proposition 7, (i); see [GS09]).

As shown by the following proposition, counterexamples may be obtained with \overline{E} a small perturbation of the hexagonal rank-two lattice¹⁸ A_2 :

Proposition 4.3.1. *For any $\lambda \in]0, 4[$, let \overline{E}_{λ} be the euclidean lattice defined by $E_{\lambda} = \mathbb{Z}^2$ inside $E_{\lambda, \mathbb{R}} = \mathbb{R}^2$ equipped with the euclidean norm $\|\cdot\|_{\lambda}$ such that*

$$(4.29) \quad \|(x, y)\|_{\lambda}^2 := \lambda(x^2 - xy) + y^2,$$

and let \overline{F}_{λ} be the sub-euclidean lattice of \overline{E}_{λ} defined by the \mathbb{Z} -submodule $F_{\lambda} := \mathbb{Z} \oplus \{0\}$ of \mathbb{Z}^2 .

Then, for any $\lambda \in]0, 4[$, we have:

$$(4.30) \quad h_{\text{Ar}}^0(\overline{F}_{\lambda}) = 0 \iff \lambda > 1,$$

$$(4.31) \quad h_{\text{Ar}}^0(\overline{E}_{\lambda}/\overline{F}_{\lambda}) \leq \log 3 \iff \lambda < 3,$$

and

$$(4.32) \quad h_{\text{Ar}}^0(\overline{E}_{\lambda}) \geq \log 5.$$

Indeed, the euclidean lattice \overline{E}_1 is nothing but the hexagonal lattice A_2 , and (4.30)–(4.32) show that, for any $\lambda \in]1, 3[$,

$$h_{\text{Ar}}^0(\overline{E}_{\lambda}) > h_{\text{Ar}}^0(\overline{F}_{\lambda}) + h_{\text{Ar}}^0(\overline{E}_{\lambda}/\overline{F}_{\lambda}).$$

¹⁸Recall that A_2 is defined as the lattice $\mathbb{Z} + \mathbb{Z}e^{2\pi i/3}$ inside \mathbb{C} (equipped with its usual absolute value), or equivalently as the lattice \mathbb{Z}^2 inside \mathbb{R}^2 equipped with the euclidean norm $\|\cdot\|_{A_2}$ defined by $\|(x, y)\|_{A_2}^2 := x^2 - xy + y^2$.

Proof of Proposition 4.3.1. Observe that the definition (4.29) of $\|\cdot\|_\lambda$ may also be written

$$(4.33) \quad \|(x, y)\|_\lambda^2 = \lambda(x - y/2)^2 + (1 - \lambda/4)y^2.$$

The equivalence (4.30) follows from the fact that \overline{F}_λ may be identified with the lattice \mathbb{Z} inside \mathbb{R} equipped with the norm $\|\cdot\|$ such that $\|1\|^2 = \lambda$.

To establish (4.31), observe that, as shown by (4.33), the orthogonal complement of $F_{\lambda, \mathbb{R}}$ in the euclidean vector space $(E_{\lambda, \mathbb{R}}, \|\cdot\|_\lambda)$ is the real line $\mathbb{R}(1/2, 1)$. Consequently, the norm in $\overline{E}_\lambda/\overline{F}_\lambda$ of the class $[(0, 1)]$ of $(0, 1)$ is given by

$$\|[(0, 1)]\|_{\overline{E}_\lambda/\overline{F}_\lambda}^2 = \|(1/2, 1)\|_{E_\lambda}^2 = 1 - \lambda/4.$$

Therefore $h_{\text{Ar}}^0(\overline{E}_\lambda/\overline{F}_\lambda) \leq \log 3$ if and only if $\sqrt{1 - \lambda/4} > 1/2$, that is, if and only if $\lambda < 3$.

Finally, the unit ball of \overline{E}_λ always contains the five lattice points $(0, 0)$, $(0, 1)$, $(0, -1)$, $(1, 1)$, and $(-1, -1)$. This proves (4.32). \square

4.4. The asymptotic invariant $\tilde{h}_{\text{Ar}}^0(\overline{E}, t)$. In a vein related to the discussion of subadditive invariants of euclidean lattices in the previous section — notably to the the discussion of Blichfeldt pairs in 4.3.2 — it may be worth mentioning that h_{Ar}^0 satisfies a *superadditivity* property, that turns out to lead to another interpretation of the θ -invariants of euclidean lattices and to relate h_{Ar}^0 and h_θ^0 to the thermodynamic formalism.

4.4.1. The invariants $h_{\text{Ar}}^0(\overline{E}, t)$ and $\tilde{h}_{\text{Ar}}^0(\overline{E}, t)$. To formulate the superadditivity of h_{Ar}^0 , it is convenient to introduce a simple generalization of this invariant. Namely, for any euclidean lattice $\overline{E} = (E, \|\cdot\|)$ and any positive real number t , we let:

$$(4.34) \quad \begin{aligned} h_{\text{Ar}}^0(\overline{E}, t) &:= \log |\{v \in E \mid \|v\|^2 \leq t\}| \\ &= h_{\text{Ar}}^0(\overline{E} \otimes \overline{\mathcal{O}}((\log t)/2)). \end{aligned}$$

The following observation is straightforward:

Lemma 4.4.1. *For any two euclidean lattices $\overline{E}_1 = (E_1, \|\cdot\|_1)$ and $\overline{E}_2 = (E_2, \|\cdot\|_2)$, and any two positive real numbers t_1 and t_2 , we have:*

$$(4.35) \quad h_{\text{Ar}}^0(\overline{E}_1, t_1) + h_{\text{Ar}}^0(\overline{E}_2, t_2) \leq h_{\text{Ar}}^0(\overline{E}_1 \oplus \overline{E}_2, t_1 + t_2).$$

Indeed, it follows from the inclusion:

$$\{v_1 \in E_1 \mid \|v_1\|_1^2 \leq t_1\} \times \{v_2 \in E_2 \mid \|v_2\|_2^2 \leq t_2\} \subset \{v \in E_1 \oplus E_2 \mid \|v\|_{E_1 \oplus E_2}^2 \leq t_1 + t_2\}.$$

\square

In particular, for any euclidean lattice \overline{E} , the sequence $(h_{\text{Ar}}^0(\overline{E}^{\oplus n}, nt))_{n \geq 1}$ is superadditive; namely, it satisfies, for every $(n_1, n_2) \in \mathbb{N}_{\geq 1}^2$:

$$(4.36) \quad h_{\text{Ar}}^0(\overline{E}^{\oplus n_1}, n_1 t) + h_{\text{Ar}}^0(\overline{E}^{\oplus n_2}, n_2 t) \leq h_{\text{Ar}}^0(\overline{E}^{\oplus (n_1 + n_2)}, (n_1 + n_2)t).$$

Besides, this sequence grows at most linearly with n :

Lemma 4.4.2. *For any euclidean lattice, when n goes to $+\infty$,*

$$(4.37) \quad h_{\text{Ar}}^0(\overline{E}^{\oplus n}, nt) = O(n).$$

Proof. For any euclidean lattice $\overline{V} := (V, \|\cdot\|)$ and for any $(P, r) \in V_{\mathbb{R}} \times \mathbb{R}_+$, we shall denote by $\overset{\circ}{B}_{\overline{V}}(P, r)$ the open ball of center P and radius r in the normed vector space $(V_{\mathbb{R}}, \|\cdot\|)$.

We shall also denote by v_n the volume of the n -dimensional unit ball. Recall that

$$v_n = \pi^{n/2} / \Gamma((n/2) + 1)$$

and that consequently, when n goes to $+\infty$,

$$(4.38) \quad \log v_n = -(n/2) \log n + O(n).$$

Let λ be the first minimum of \overline{E} . Observe that, for any two points v and w of $E^{\oplus n}$:

$$v \neq w \implies \left(v + \mathring{B}_{\overline{E}}(0, \lambda/2)^n \right) \cap \left(w + \mathring{B}_{\overline{E}}(0, \lambda/2)^n \right) = \emptyset.$$

(Compare with the proof of Lemma 3.6.3.)

Besides, if $\mathcal{E}(n, t) := \left\{ v \in E^{\oplus n} \mid \|v\|_{\overline{E}^{\oplus n}}^2 \leq nt \right\}$, then we have:

$$(4.39) \quad \bigcup_{v \in \mathcal{E}(n, t)} \left(v + \mathring{B}_{\overline{E}}(0, \lambda/2)^n \right) \subset \bigcup_{v \in \mathcal{E}(n, t)} \mathring{B}_{\overline{E}^{\oplus n}}(v, \lambda\sqrt{n}/2) \subset \mathring{B}_{\overline{E}^{\oplus n}}(0, \sqrt{n}(\sqrt{t} + \lambda/2)).$$

If $e := \text{rk } E$, we finally obtain, by considering the Lebesgue measures of the first and last sets in (4.39):

$$(4.40) \quad v_e^n |\mathcal{E}(n, t)| \leq v_{ne} [\sqrt{n}(\sqrt{t} + \lambda/2)]^{ne}.$$

Finally, when n goes to $+\infty$, from (4.40) and (4.38), we obtain:

$$h_{\text{Ar}}^0(\overline{E}^{\oplus n}, nt) = \log |\mathcal{E}(n, t)| \leq ne \log \sqrt{n} + \log v_{ne} + O(n) = O(n).$$

□

Recall that, according to a well-known observation that goes back to Fekete [Fek23], superadditive sequences of real numbers have a simple asymptotic behaviour:

Lemma 4.4.3. *Let $(a_n)_{n \in \mathbb{N}_{\geq 1}}$ be a sequence of real numbers that is superadditive (that is, such that $a_{n_1+n_2} \geq a_{n_1} + a_{n_2}$ for any $(n_1, n_2) \in \mathbb{N}_{\geq 1}^2$).*

Then the sequence $(a_n/n)_{n \in \mathbb{N}_{\geq 1}}$ admits a limit in $]-\infty, +\infty]$. Moreover:

$$\lim_{n \rightarrow +\infty} a_n/n = \sup_{n \in \mathbb{N}_{\geq 1}} a_n/n.$$

□

Together with the superadditivity property (4.36) and with the bound (4.37), Fekete's Lemma establishes the following

Proposition 4.4.4. *For any euclidean lattice \overline{E} and any $t \in \mathbb{R}_+^*$, the limit*

$$\tilde{h}_{\text{Ar}}^0(\overline{E}, t) := \lim_{n \rightarrow +\infty} \frac{1}{n} h_{\text{Ar}}^0(\overline{E}^{\oplus n}, nt)$$

exists in \mathbb{R}_+ . Moreover, we have:

$$\tilde{h}_{\text{Ar}}^0(\overline{E}, t) := \sup_{n \in \mathbb{N}_{\geq 1}} \frac{1}{n} h_{\text{Ar}}^0(\overline{E}^{\oplus n}, nt).$$

□

The so-defined invariant $\tilde{h}_{\text{Ar}}^0(\overline{E}, t)$ is an “asymptotic version” of the more naive invariant $h_{\text{Ar}}^0(\overline{E}, t)$. By its very definition, it satisfies, for every positive integer k ,

$$\tilde{h}_{\text{Ar}}^0(\overline{E}^{\oplus k}, kt) = k \tilde{h}_{\text{Ar}}^0(\overline{E}, t),$$

and inherits the superadditivity property of h_{Ar}^0 stated in Lemma 4.4.1; namely, with the notation of *loc. cit.*, we have:

$$(4.41) \quad \tilde{h}_{\text{Ar}}^0(\overline{E}_1, t_1) + \tilde{h}_{\text{Ar}}^0(\overline{E}_2, t_2) \leq \tilde{h}_{\text{Ar}}^0(\overline{E}_1 \oplus \overline{E}_2, t_1 + t_2).$$

4.4.2. *The invariant $\tilde{h}_{\text{Ar}}^0(\overline{E}, t)$ and the Legendre transform of $\log \theta_{\overline{E}}$.* The invariant $\tilde{h}_{\text{Ar}}^0(\overline{E}, t)$ attached to some euclidean lattice \overline{E} , as a function of $t \in \mathbb{R}_+^*$, turns out to be simply related to the theta function $\theta_{\overline{E}}$ of \overline{E} , and consequently to enjoy various properties — notably, it is a real analytic function — that are not obvious on its original definition.

To express this relation, recall that, provided \overline{E} has positive rank, the function

$$\log \theta_{\overline{E}} : \mathbb{R}_+^* \longrightarrow \mathbb{R}_+^*$$

is a decreasing real analytic diffeomorphism, that moreover is *convex*. Actually the function

$$U_{\overline{E}} := -(\log \theta_{\overline{E}})'$$

satisfies, for every $\beta \in \mathbb{R}_+^*$,

$$U_{\overline{E}}(\beta) = \frac{\sum_{v \in E} \pi \|v\|^2 e^{-\beta \pi \|v\|^2}}{\sum_{v \in E} e^{-\beta \pi \|v\|^2}},$$

and is easily seen to establish a decreasing real analytic diffeomorphism

$$U_{\overline{E}} : \mathbb{R}_+^* \xrightarrow{\sim} \mathbb{R}_+^*.$$

Theorem 4.4.5. *For every euclidean lattice \overline{E} of positive rank, the function $\tilde{h}_{\text{Ar}}^0(\overline{E}, \cdot)$ is real analytic, increasing, concave and surjective from \mathbb{R}_+^* to \mathbb{R}_+^* .*

Moreover, if we let:

$$(4.42) \quad S_{\overline{E}}(x) := \tilde{h}_{\text{Ar}}^0(\overline{E}, x/\pi) \quad (x \in \mathbb{R}_+^*),$$

then the functions $-S_{\overline{E}}(-\cdot)$ and $\log \theta_{\overline{E}}$ are Legendre transforms of each other.

Namely, for every $x \in \mathbb{R}_+^*$,

$$(4.43) \quad \tilde{h}_{\text{Ar}}^0(\overline{E}, x) = \inf_{\beta > 0} (\pi \beta x + \log \theta_{\overline{E}}(\beta))$$

and, for every $\beta \in \mathbb{R}_+^*$,

$$(4.44) \quad \log \theta_{\overline{E}}(\beta) = \sup_{x > 0} (\tilde{h}_{\text{Ar}}^0(\overline{E}, x) - \pi \beta x).$$

Moreover the derivative $S'_{\overline{E}}$ establishes a real analytic decreasing diffeomorphism

$$S'_{\overline{E}} : \mathbb{R}_+^* \xrightarrow{\sim} \mathbb{R}_+^*.$$

inverse of $U_{\overline{E}}$, and for every $x \in \mathbb{R}_+^*$, the infimum in the right-hand side of (4.43) is attained for a unique value β , namely for

$$\beta = S'_{\overline{E}}(\pi x) = \pi^{-1} \tilde{h}_{\text{Ar}}^0(\overline{E}, \cdot)'(x).$$

Dually, for every $\beta \in \mathbb{R}_+^*$, the supremum in the right-hand side of (4.44) is attained for a unique value of x , namely for $x = \pi^{-1} U_{\overline{E}}(\beta)$.

When \overline{E} is the “trivial” euclidean lattice of rank one $\overline{\mathcal{O}}(0) := (\mathbb{Z}, |\cdot|)$, Theorem 4.4.5 may be deduced from results of Mazo and Odlyzko ([MO90], Theorem 1).

In its general formulation above, Theorem 4.4.5 is a consequence of the extension of Cramér’s theory of large deviations presented in Appendix A, concerning an arbitrary measure space $(\mathcal{E}, \mathcal{T}, \mu)$ and a non-negative measurable function H on \mathcal{E} . Theorem 4.4.5 gathers the results of Paragraph A.5.1 (notably Theorem A.5.1 and Corollary A.5.2) in the situation where

$$(\mathcal{E}, \mathcal{T}, \mu) = (E, \mathcal{P}(E), \sum_{v \in E} \delta_v)$$

— that is, the set E underlying the euclidean lattice \overline{E} equipped with the counting measure — and where

$$H := \pi \|\cdot\|^2.$$

Indeed, in this situation, the function Ψ introduced in paragraph A.5.1 is nothing else than $\log \theta_{\overline{E}}$ and the function S is simply the function $S_{\overline{E}}$ defined by (4.42). One may also observe that, specialized to this situation, the arguments in Appendix A actually provide an alternative proof of the finiteness of $\tilde{h}_{\text{Ar}}^0(\overline{E}, t)$, that relies on the properties of the theta functions of lattices and avoids the estimates in the proof of Lemma 4.4.2.

The above construction, of data $(\mathcal{E}, \mathcal{T}, \mu)$ and H of the type considered in Appendix A from euclidean lattices, is clearly compatible with finite products: the data associated to the direct sum $\bigoplus_{i \in I} \overline{E}_i$ of a finite family $(\overline{E}_i)_{i \in I}$ of euclidean lattices may be identified with the “product”, in the sense of Paragraph A.5.2, of the data associated to each of the \overline{E}_i .

This observation allows us to apply Proposition A.5.3 to analyze the invariants \tilde{h}_{Ar}^0 of a direct sum of euclidean lattices. Notably, we immediately obtain the following more precise form of the superadditivity (4.41) of \tilde{h}_{Ar}^0 :

Proposition 4.4.6. *For any two euclidean lattices of positive rank \overline{E}_1 and \overline{E}_2 and any $t \in \mathbb{R}_+^*$,*

$$(4.45) \quad \tilde{h}_{\text{Ar}}^0(\overline{E}_1 \oplus \overline{E}_2, t) = \max_{\substack{t_1, t_2 > 0 \\ t_1 + t_2 = t}} \left(\tilde{h}_{\text{Ar}}^0(\overline{E}_1, t_1) + \tilde{h}_{\text{Ar}}^0(\overline{E}_2, t_2) \right).$$

Moreover the maximum is attained for a unique pair (t_1, t_2) , namely for $(\pi^{-1}U_{\overline{E}_1}(\beta), \pi^{-1}U_{\overline{E}_2}(\beta))$ where $\beta := S'_{\overline{E}_1 \oplus \overline{E}_2}(\pi t)$. \square

This proposition can be understood as an expression of the second law of thermodynamics in the context of euclidean lattices. As a last element pleading for an interpretation of the θ -invariants of euclidean lattices in terms of statistical thermodynamics, let us briefly translate to the framework of this section the results in the last paragraph A.5.4 of Appendix A.

Let us consider denote an euclidean lattice of positive rank $\overline{E} := (E, \|\cdot\|)$ and let us denote by

$$\mathcal{C} := \{p \in [0, 1]^E \mid \sum_{e \in E} p(e) = 1\}$$

the (compact convex) space of probability measures on E .

We may consider the functions ϵ (“energy”) and I (“information theoretic entropy”) from \mathcal{C} to $[0, +\infty]$ defined as follows:

$$\epsilon(p) := \sum_{e \in E} p(e) \pi \|e\|^2$$

and

$$I(p) := - \sum_{e \in E} p(e) \log p(e).$$

Then Proposition A.5.5, applied to $(\mathcal{E}, \mathcal{T}, \mu) = (E, \mathcal{P}(E), \sum_{v \in E} \delta_v)$ and $H = \pi \|\cdot\|^2$, becomes the following statement:

Proposition 4.4.7. *Let u and β be two positive real numbers such that $u = U_{\overline{E}}(\beta)$.*

For any p in \mathcal{C} such that $\epsilon(p) = u$, we have:

$$(4.46) \quad I(p) \leq S_{\overline{E}}(\beta).$$

Moreover the equality is achieved in (4.46) for a unique p in $\epsilon^{-1}(u)$, namely for the measure p_β defined by

$$p_\beta(e) := \theta_{\overline{E}}(\beta)^{-1} e^{-\pi \beta \|e\|^2}.$$

\square

4.4.3. Duality and further comparison estimates. In this paragraph, we denote by \overline{E} an euclidean lattice of positive rank, and we derive additional estimates comparing its invariants $h_{\text{Ar}}^0(\overline{E}, \cdot)$, $\tilde{h}_{\text{Ar}}^0(\overline{E}, \cdot)$ and $h_\theta^0(\overline{E})$.

Ultimately, these estimates will appear as consequences of (i) the “thermodynamic” formalism of the previous paragraph and (ii) the Poisson formula (3.37), which relates the theta functions $\theta_{\overline{E}}$ and $\theta_{\overline{E}^\vee}$ of \overline{E} and of its dual euclidean lattice \overline{E}^\vee .

Indeed, from (3.37), we immediately get:

Proposition 4.4.8. *For any $\beta \in \mathbb{R}_+^*$, we have:*

$$(4.47) \quad \log \theta_{\overline{E}}(\beta) - \log \theta_{\overline{E}^\vee}(\beta^{-1}) = -(\text{rk } E/2) \log \beta + \widehat{\deg} \overline{E},$$

$$(4.48) \quad \beta U_{\overline{E}}(\beta) + \beta^{-1} U_{\overline{E}^\vee}(\beta^{-1}) = \text{rk } E/2,$$

and

$$(4.49) \quad 0 \leq U_{\overline{E}}(\beta) \leq \text{rk } E/(2\beta).$$

□

The expression $\text{rk } E/(2\beta)$ which appears in the upper-bound on $U_{\overline{E}}(\beta)$ in (4.49) coincides with the function $U(\beta)$ in the “Maxwellian” situation discussed in Paragraph A.5.3. This upper-bound was also the key point behind the estimates *à la Banaszczyk* derived in Lemma 4.1.5.

For any integer $n \geq 1$, we let:

$$C(n) := -\sup_{t>1} [\log(1-t^{-1}) - (n/2) \log t].$$

One easily shows that

$$C(n) = \log(n/2) + (1+n/2) \log(1+2/n)$$

and that

$$1 \leq C(n) - \log(n/2) \leq (3/2) \log 3.$$

Theorem 4.4.9. *For any $x \in \mathbb{R}_+^*$, we have:*

$$(4.50) \quad \log \theta_{\overline{E}}(\text{rk } E/(2\pi x)) \leq h_{\text{Ar}}^0(\overline{E}, x) + C(\text{rk } E),$$

$$(4.51) \quad \log \theta_{\overline{E}}(\text{rk } E/(2\pi x)) \leq \tilde{h}_{\text{Ar}}^0(\overline{E}, x),$$

and

$$(4.52) \quad \tilde{h}_{\text{Ar}}^0(\overline{E}, x) \leq \log \theta_{\overline{E}}(\text{rk } E/(2\pi x)) + \text{rk } E/2.$$

The following consequence of Theorem 4.4.9, which involves only the “elementary” invariants $h_{\text{Ar}}^0(\overline{E}, t)$ and $\tilde{h}_{\text{Ar}}^0(\overline{E}, t)$, seems worth being mentioned:

Corollary 4.4.10. *For any $x \in \mathbb{R}_+^*$,*

$$(4.53) \quad 0 \leq \tilde{h}_{\text{Ar}}^0(\overline{E}, x) - h_{\text{Ar}}^0(\overline{E}, x) \leq C(\text{rk } E) + \text{rk } E/2.$$

□

Proof of Theorem 4.4.9. Let us start with a straightforward consequence of Lemma 4.1.5 (after the change of notation: $x = r^2$ and $\beta = t$):

Lemma 4.4.11. *For any $(x, \beta) \in \mathbb{R}_+^{*2}$ such that $\beta x > \text{rk } E/(2\pi)$,*

$$(4.54) \quad h_\theta^0(\overline{E}, x) \geq \log(1 - \text{rk } E/(2\pi\beta x)) + \log \theta_{\overline{E}}(\beta).$$

□

From Lemma 4.4.11, we easily deduce:

Lemma 4.4.12. *For any $(x, \beta, \beta') \in \mathbb{R}_+^{*3}$ such that $\beta x \geq \text{rk } E/(2\pi)$ and $\beta' > \beta$,*

$$(4.55) \quad h_\theta^0(\overline{E}, x) \geq \log(1 - \text{rk } E/(2\pi\beta'x)) - (\text{rk } E/2) \log(\beta'/\beta) + \log \theta_{\overline{E}}(\beta).$$

Proof. Lemma 4.4.11 implies that

$$h_\theta^0(\overline{E}, x) \geq \log(1 - \text{rk } E/(2\pi\beta'x)) + \log \theta_{\overline{E}}(\beta').$$

Besides, we have:

$$(4.56) \quad \log \theta_{\overline{E}}(\beta') \geq \log \theta_{\overline{E}}(\beta) - (\text{rk } E/2) \log(\beta'/\beta).$$

This follows for instance from the upper-bound in (4.49) on $U_{\overline{E}}(\beta) := -(\log \theta_{\overline{E}})'(\beta)$. \square

To prove the inequality (4.50), we apply Lemma 4.55 with $\beta := \text{rk } E/(2\pi x)$ and we observe that

$$(4.57) \quad \sup_{\beta' \in]\beta, +\infty[} [\log(1 - \text{rk } E/(2\pi\beta'x)) - (\text{rk } E/2) \log(\beta'/\beta)] \\ = \sup_{\beta' \in]\beta, +\infty[} [\log(1 - (\beta'/\beta)^{-1}) - (\text{rk } E/2) \log(\beta'/\beta)] = -C(\text{rk } E).$$

For any positive integer n , from (4.50) applied to $E^{\oplus n}$ and to nx instead of \overline{E} and x , we get:

$$n \log \theta_{\overline{E}}(\text{rk } E/(2\pi x)) \leq \tilde{h}_{\text{Ar}}^0(\overline{E}^{\oplus n}, nx) + C(n \text{rk } E).$$

Multiplying by $1/n$ and letting n go to $+\infty$, we obtain (4.51), since $C(n \text{rk } E) = o(n)$.

An alternative proof of (4.51) consists in observing that, as an easy consequence of (4.56) (which holds when $\beta' \geq \beta$), the following inequality holds for any $(\beta, \beta') \in \mathbb{R}_+^{*2}$:

$$(\text{rk } E/2)(\beta'/\beta) + \log \theta_{\overline{E}}(\beta') \geq \log \theta_{\overline{E}}(\beta).$$

Then (4.51) follows from the expression (4.43) of $\tilde{h}_{\text{Ar}}^0(\overline{E}, \cdot)$ in terms of the Legendre transform of $\log \theta_{\overline{E}}$.

Finally (4.50) follows from (4.43) or (4.44), by choosing x and β related by $\beta x = \text{rk } E/(2\pi)$. \square

The estimates in Theorem 4.4.9 show that the expressions $h_{\text{Ar}}^0(\overline{E}, \text{rk } E/2\pi)$ and $\tilde{h}_{\text{Ar}}^0(\overline{E}, \text{rk } E/2\pi)$ satisfy:

$$(4.58) \quad -C(\text{rk } E) \leq h_{\text{Ar}}^0(\overline{E}, \text{rk } E/2\pi) - h_\theta^0(\overline{E}) \leq \text{rk } E/2$$

and

$$(4.59) \quad 0 \leq \tilde{h}_{\text{Ar}}^0(\overline{E}, \text{rk } E/2\pi) - h_\theta^0(\overline{E}) \leq \text{rk } E/2.$$

These comparison estimates, relating Arakelov and θ -invariants of euclidean lattices, should be compared with the comparison estimate (4.2) in Section 4.1. The error term, of the order of $(1/2)\text{rk } E \cdot \log \text{rk } E$ in (4.2), is replaced by $\text{rk } E/2$ in (4.58) and (4.59). (These error terms will be shown to be basically optimal when $\text{rk } E$ goes to $+\infty$ in the next section; see paragraph 4.5.3 *infra*.)

These remarks plead for considering the positive real numbers

$$h_{\text{Ar}}^0(\overline{E}, \text{rk } E/2\pi) \text{ and } \tilde{h}_{\text{Ar}}^0(\overline{E}, \text{rk } E/2\pi)$$

attached to some euclidean lattice of positive rank \overline{E} as variants of

$$h_{\text{Ar}}^0(\overline{E}) = h_{\text{Ar}}^0(\overline{E}, 1)$$

that are “better behaved” than $h_{\text{Ar}}^0(\overline{E})$ itself.

4.5. Some consequences of Siegel’s mean value theorem.

4.5.1. *Siegel's mean value theorem over $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ and over $\mathcal{L}(n, \delta)$.* In this paragraph, we denote by n an integer ≥ 2 .

The Lie group $SL_n(\mathbb{R})$ is unimodular and its discrete subgroup $SL_n(\mathbb{Z})$ has a finite covolume. We shall denote by μ_n the Haar measure on $SL_n(\mathbb{R})$ which satisfies the following normalization condition: *the measure induced by μ_n on the quotient $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ — that we shall still denote by μ_n — is a probability measure.* In other words,

$$\int_{SL_n(\mathbb{R})/SL_n(\mathbb{Z})} d\mu_n = 1.$$

For any Borel function $\phi : \mathbb{R}^n \rightarrow [0, +\infty]$ and any $g \in SL_n(\mathbb{R})$, we may consider the sum

$$\Sigma(\phi)(g) := \sum_{v \in \mathbb{Z}^n \setminus \{0\}} \phi(g.v).$$

Clearly, for any $\gamma \in SL_n(\mathbb{Z})$, we have

$$\Sigma(\phi)(g.\gamma) = \Sigma(\phi)(g)$$

and the function

$$\Sigma(\phi) : SL_n(\mathbb{R})/SL_n(\mathbb{Z}) \rightarrow [0, +\infty]$$

so defined is a Borel function.

In its most basic form, Siegel's mean value theorem is the following statement ([Sie45]; see also [Wei46] and [MR58] for other derivations, and [Wei82], Chapter III, for a “modern” presentation).

Theorem 4.5.1. *For any Borel function $\phi : \mathbb{R}^n \rightarrow [0, +\infty]$ as above, the following equality holds:*

$$(4.60) \quad \int_{SL_n(\mathbb{R})/SL_n(\mathbb{Z})} \Sigma(\phi)(g) d\mu_n(g) = \int_{\mathbb{R}^n} \phi(v) d\lambda_n(v),$$

where we denote by λ_n the Lebesgue measure on \mathbb{R}^n . □

For any $g \in SL_n(\mathbb{R})$ and any $\delta \in \mathbb{R}$, the lattice $e^{-\delta/n}g(\mathbb{Z}^n)$ in \mathbb{R}^n equipped with the standard euclidean norm $\|\cdot\|_n$ (defined by $\|(x_1, \dots, x_n)\|^2 = x_1^2 + \dots + x_n^2$) becomes an euclidean lattice $(e^{-\delta/n}g(\mathbb{Z}^n), \|\cdot\|_n)$ of covolume $e^{-\delta}$, or equivalently, of Arakelov degree δ .

We shall denote by $\mathcal{L}(n, \delta)$ the set of isomorphism classes of euclidean lattices of rank n and Arakelov degree δ . This set may be endowed with a natural locally compact topology (actually, with a structure of “orbifold”) by means of the identification of the set

$$\mathcal{L}(n) := \prod_{\delta \in \mathbb{R}} \mathcal{L}(n, \delta)$$

of isomorphism classes of euclidean lattices of rank n with the double coset space

$$O_n(\mathbb{R}) \backslash GL_n(\mathbb{R})/SL_n(\mathbb{Z}).$$

In concrete terms, the natural topology and Borel structures on $\mathcal{L}(n, \delta)$ are the quotients of the ones of $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ by the surjective map

$$\begin{aligned} \pi_{n,\delta} : SL_n(\mathbb{R})/SL_n(\mathbb{Z}) &\longrightarrow \mathcal{L}_{n,\delta} \\ [g] &\longmapsto [(e^{-\delta/n}g(\mathbb{Z}^n), \|\cdot\|_n)]. \end{aligned}$$

We shall denote by

$$\mu_{n,\delta} := \pi_{n,\delta*} \mu_n$$

the Borel measure on $\mathcal{L}_{n,\delta}$ deduced from the measure μ_n on $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ by the parametrization $\pi_{n,\delta}$ of $\mathcal{L}_{n,\delta}$. Like μ_n , it is a probability measure:

$$\int_{\mathcal{L}(n,\delta)} d\mu_{n,\delta} = 1.$$

Applied to a radial function $\phi : \mathbb{R}^n \rightarrow [0, +\infty]$, Siegel's mean value formula (4.60) “descends” through $\pi_{n,\delta}$.

Namely, to any Borel function

$$\rho : \mathbb{R}_+^* \rightarrow [0, +\infty]$$

we may attach the Borel function

$$\Sigma_n(\rho) : \mathcal{L}(n) \rightarrow [0, +\infty]$$

which maps the isomorphism class of some euclidean lattice $\overline{E} := (E, \|\cdot\|)$ of rank n to

$$\Sigma_n(\rho)(\overline{E}) := \sum_{v \in E \setminus \{0\}} \rho(\|v\|).$$

For any $\delta \in \mathbb{R}$ and any $g \in SL_n(\mathbb{R})$, we have:

$$\Sigma_n(\rho)(\pi_{n,\delta}([g])) = \sum_{w \in \mathbb{Z}^k \setminus \{0\}} \rho(e^{-\delta/n} \|g.w\|) = \Sigma(\phi_\delta)([g])$$

where ϕ_δ is the function from \mathbb{R}^n to $[0, +\infty]$ defined by:

$$\phi_\delta(v) := \rho(e^{-\delta/n} \|v\|).$$

Clearly, we have

$$\int_{\mathbb{R}^n} \phi_\delta(v) d\lambda_n(v) = e^\delta \int_{\mathbb{R}^n} \rho(\|v\|) d\lambda_n(v),$$

and Siegel's mean value formula (4.60) applied to ϕ_δ becomes:

Theorem 4.5.2. *For any $(n, \delta) \in \mathbb{N}_{\geq 2} \times \mathbb{R}$ and for any Borel function $\rho : \mathbb{R}^+ \rightarrow [0, +\infty]$, the following equality holds:*

$$(4.61) \quad \int_{\mathcal{L}(n,\delta)} \Sigma_n(\rho) d\mu_{n,\delta} = e^\delta \int_{\mathbb{R}^n} \rho(\|v\|_n) d\lambda_n(v).$$

□

The last integral may also be written

$$\int_{\mathbb{R}^n} \rho(\|v\|_n) d\lambda_n(v) = nv_n \int_0^{+\infty} \rho(r) r^{n-1} dr$$

where, as previously in this article, v_n denotes the volume of the n -dimensional ball:

$$v_n := \lambda_n(\{v \in \mathbb{R}^n \mid \|v\|_n < 1\}) = \frac{\pi^{n/2}}{\Gamma(1 + n/2)}.$$

Theorems 4.5.1 and 4.5.2 are classically used to establish the existence of euclidean lattices satisfying suitable conditions — for instance, of lattices of large enough density — by “probabilistic arguments”, based on the observation that a positive measurable function on some probability space assume values greater or equal to its mean value on some subset of positive measure. We refer the reader to [Sie45] for a concise discussion of the existence of “dense lattices” as a consequence of Theorem 4.5.1 and for references to related earlier work of Minkowski and Hlawka.

For later reference, we state a formal version of the above observation as the following lemma:

Lemma 4.5.3. *Let ϕ and ψ be two Borel functions from $\mathcal{L}(n, \delta)$ to $[0, +\infty]$.*

1) If the integrals

$$I_\phi := \int_{\mathcal{L}(n,\delta)} \phi(x) d\mu_{n,\delta}(x) \text{ and } I_\psi := \int_{\mathcal{L}(n,\delta)} \psi(x) d\mu_{n,\delta}(x)$$

are finite and positive, then the Borel subsets

$$\mathcal{E}_{\leq} := \{x \in \mathcal{L}(n, \delta) \mid \phi(x)/I_{\phi} \leq \psi(x)/I_{\psi}\}$$

and

$$\mathcal{E}_{\geq} := \{x \in \mathcal{L}(n, \delta) \mid \phi(x)/I_{\phi} \geq \psi(x)/I_{\psi}\}$$

have positive $\mu_{n,\delta}$ -measures, and therefore are non-empty.

2) In particular, if the integral I_{ϕ} is finite, there exists x in $\mathcal{L}(n, \delta)$ such that

$$\phi(x) \leq I_{\phi}.$$

If moreover ϕ is continuous and non constant, then I_{ϕ} is positive and the image $\phi(\mathcal{L}(n, \delta))$ contains an open neighborhood of I_{ϕ} in \mathbb{R}_{+}^{*} . \square

4.5.2. Applications to h_{Ar}^0 and h_{θ}^0 . Let us start by recovering a simple version of the classical results of Minkowski-Hlawka-Siegel alluded to above. We will express it in terms of the invariant $h_{\text{Ar}}^0(., t)$, in a form convenient for later references and for comparison with similar results concerning the invariant h_{θ}^0 .

According to the very definition of $h_{\text{Ar}}^0(., t)$, we have

$$e^{h_{\text{Ar}}^0(\overline{E}, t)} - 1 = |\{v \in E \setminus \{0\} \mid \|v\|^2 \leq t\}| = \sum_{v \in E \setminus \{0\}} \mathbf{1}_{[0, t^{1/2}]}(\|v\|) = \Sigma_n(\mathbf{1}_{[0, t^{1/2}]})([\overline{E}]).$$

Besides, for $\rho = \mathbf{1}_{[0, t^{1/2}]}$, the computation of the integral in the right-hand side of Siegel's mean value formula (4.61) is straightforward — indeed,

$$\int_{\mathbb{R}^n} \mathbf{1}_{[0, t^{1/2}]}(\|v\|) d\lambda_n(v) = v_n \cdot t^{n/2}$$

— and formula (4.61) takes the following form:

Proposition 4.5.4. *For any (n, δ) in $\mathbb{N}_{\geq 2} \times \mathbb{R}$ and any $t \in \mathbb{R}_{+}^{*}$, the following relations hold:*

$$(4.62) \quad \int_{[\overline{E}] \in \mathcal{L}(n, \delta)} e^{h_{\text{Ar}}^0(\overline{E}, t)} d\mu_{n, \delta}([\overline{E}]) = 1 + v_n t^{n/2} e^{\delta}.$$

In particular, when $t = 1$, we obtain:

$$\int_{[\overline{E}] \in \mathcal{L}(n, \delta)} e^{h_{\text{Ar}}^0(\overline{E})} d\mu_{n, \delta}([\overline{E}]) = 1 + v_n e^{\delta}.$$

Observe that $e^{h_{\text{Ar}}^0(., t)} - 1$ takes its values in \mathbb{N} and therefore vanishes where it is < 1 . Therefore, if we apply Lemma 4.5.3, part 2), to the function $\phi := e^{h_{\text{Ar}}^0(., t)} - 1$, we obtain that, for any (n, δ) in $\mathbb{N}_{\geq 2} \times \mathbb{R}$ and any $t \in \mathbb{R}_{+}^{*}$ such that

$$v_n t^{n/2} e^{\delta} < 1,$$

there exists some euclidean lattice \overline{E} of rank n and Arakelov degree δ such that $h_{\text{Ar}}^0(\overline{E}, t) = 0$, or equivalently, such that $\lambda_1(\overline{E}) > t^{1/2}$.

In other words, we have established the following variant of a classical result of Minkowski:

Corollary 4.5.5. *For any (n, δ) in $\mathbb{N}_{\geq 2} \times \mathbb{R}$, we have:*

$$(4.63) \quad \sup_{[\overline{E}] \in \mathcal{L}(n, \delta)} \lambda_1(\overline{E}) \geq e^{-\delta/n} v_n^{-1/n}.$$

This estimates has to be compared with the upper bound

$$(4.64) \quad \sup_{[\overline{E}] \in \mathcal{L}(n, \delta)} \lambda_1(\overline{E}) \leq 2e^{-\delta/n} v_n^{-1/n}$$

that follows from the so-called “Minkowski First Theorem”.

Let us also recall that, when n goes to infinity,

$$v_n^{-1/n} \sim \sqrt{n/(2\pi e)}$$

and that the positive real number

$$\gamma_n := \sup_{[\overline{E}] \in \mathcal{L}(n,0)} \lambda_1(\overline{E})^2$$

is classically known as the *Hermite constant* in dimension n . Thus the lower-bound (4.63), when expressed in terms of Hermite constants, takes the following asymptotic form:

$$\liminf_{n \rightarrow +\infty} \gamma_n/n \geq 1/(2\pi e),$$

well-known in the study of sphere packings (see [CS93], notably Chapter 1, for additional informations and references).

To compute the average value on $\mathcal{L}(n, \delta)$ of the θ -invariants, we apply Siegel's mean value formula (4.61) to the Gaussian function

$$\rho(x) := e^{-\pi x^2}.$$

For this choice of ρ , the integral in the right-hand side of (4.61) is simply:

$$\int_{\mathbb{R}^n} e^{-\pi \|v\|^2} d\lambda_n(v) = 1,$$

and Siegel's mean value formula takes the following form:

Proposition 4.5.6. *For any (n, δ) in $\mathbb{N}_{\geq 2} \times \mathbb{R}$,*

$$(4.65) \quad \int_{[\overline{E}] \in \mathcal{L}(n, \delta)} e^{h_\theta^0(\overline{E})} d\mu_{n, \delta}([\overline{E}]) = 1 + e^\delta$$

□

This expression for the mean value of $e^{h_\theta^0(\overline{E})}$ has to be compared with the lower bound

$$h_\theta^0(\overline{E}) \geq \delta$$

valid over $\mathcal{L}(n, \delta)$ (see (3.15)).

The function on $\mathcal{L}(n, \delta)$ defined by h_θ^0 is clearly continuous. Moreover it is non-constant (this follows for instance from its expression for direct sums of rank-one euclidean lattices in Proposition 3.5.2). Therefore we may apply the last assertion of Lemma 4.5.3 to the function $e^{h_\theta^0}$ and we obtain, from the value of its integral computed in (4.65):

Corollary 4.5.7. *For any (n, δ) in $\mathbb{N}_{\geq 2} \times \mathbb{R}$, there exists an euclidean lattice \overline{E} of rank n and degree δ such that*

$$(4.66) \quad h_\theta^0(\overline{E}) < \log(1 + e^\delta).$$

□

According to the Poisson-Riemann-Roch formula, for any $[\overline{E}]$ in $\mathcal{L}(n, \delta)$, we have:

$$h_\theta^0(\overline{E}) - h_\theta^1(\overline{E}) = \delta.$$

Therefore the equality (4.65) may be also written:

$$(4.67) \quad \int_{[\overline{E}] \in \mathcal{L}(n, \delta)} e^{h_\theta^1(\overline{E})} d\mu_{n, \delta}([\overline{E}]) = 1 + e^{-\delta},$$

and the condition (4.66) is equivalent to:

$$h_\theta^1(\overline{E}) < \log(1 + e^{-\delta}).$$

To put the conclusion of Corollary 4.5.7 in perspective, we may consider the “obvious” euclidean lattice of rank n and Arakelov degree δ , namely $\overline{\mathcal{O}}(\delta/n)^{\oplus n}$, for any $(n, \delta) \in \mathbb{N}_{\geq 1} \times \mathbb{R}$. If we define, for every $t \in \mathbb{R}_+^*$,

$$\theta(t) := \theta_{\overline{\mathcal{O}}(0)}(t) = \sum_{k \in \mathbb{Z}} e^{-\pi k^2 t},$$

its θ -invariant is:

$$h_{\theta}^0(\overline{\mathcal{O}}(\delta/n)^{\oplus n}) = n \log \theta(e^{-2\delta/n}).$$

When δ is fixed and n goes to infinity, this expression is equivalent to

$$n \log \theta(0) = n h_{\theta}^0(\overline{\mathcal{O}}) = n \eta.$$

This demonstrates that the existence of a (class of) euclidean lattice in $\mathcal{L}(n, \delta)$ satisfying (4.66) is not “obvious” when n is large.

We may also apply the first part of Lemma 4.5.3 to the functions $e^{h_{\text{Ar}}^0(\cdot, t)}$ and $e^{h_{\theta}^0}$. Taking into account the expressions (4.62) and (4.65) for their integrals, we obtain:

Corollary 4.5.8. *For any $(n, \delta, t) \in \mathbb{N}_{\geq 2} \times \mathbb{R} \times \mathbb{R}_+^*$, there exist euclidean lattices \overline{E}_+ and \overline{E}_- , of rank n and Arakelov degree δ , such that*

$$h_{\text{Ar}}^0(\overline{E}_+, t) - h_{\theta}^0(\overline{E}_+) \geq \log \frac{1 + v_n t^{n/2} e^{\delta}}{1 + e^{\delta}}$$

and

$$h_{\text{Ar}}^0(\overline{E}_-, t) - h_{\theta}^0(\overline{E}_-) \leq \log \frac{1 + v_n t^{n/2} e^{\delta}}{1 + e^{\delta}}.$$

□

4.5.3. Constants in comparison estimates. From Corollary 4.5.8, one easily derives that the additive constants in diverse estimates relating the invariants $h_{\text{Ar}}^0(\overline{E}, t)$ and $h_{\theta}^0(\overline{E})$ established in the previous sections are “of the correct order of growth” when the rank of the euclidean lattice \overline{E} goes to $+\infty$.

For instance, consider the first inequality in (4.2). It asserts that, for any euclidean lattice \overline{E} of rank $n \geq 1$,

$$(4.68) \quad h_{\text{Ar}}^0(\overline{E}) - h_{\theta}^0(\overline{E}) \geq -(n/2) \cdot \log n + \log(1 - 1/2\pi).$$

According to Corollary 4.5.8 applied with $t = 1$ and $\delta = 0$, for every $n \in \mathbb{N}_{\geq 2}$, there exists an euclidean lattice of rank n such that

$$\text{covol}(\overline{E}) = 1$$

and

$$h_{\text{Ar}}^0(\overline{E}_-) - h_{\theta}^0(\overline{E}_-) \leq \log \frac{1 + v_n}{2}.$$

Besides, when n goes to $+\infty$,

$$\log \frac{1 + v_n}{2} = -(n/2) \cdot \log n + O(n).$$

This shows that the “best constant” in the right-hand side of (4.68) — even if one considers euclidean lattices of covolume 1 only — is equivalent to $-(n/2) \cdot \log n$ when n goes to $+\infty$.

Consider now the estimates, valid for any euclidean lattice \overline{E} of rank $n \geq 1$,

$$(4.69) \quad h_{\text{Ar}}^0(\overline{E}, n/2\pi) - h_{\theta}^0(\overline{E}, n/2\pi) \leq \tilde{h}_{\text{Ar}}^0(\overline{E}, n/2\pi) - h_{\theta}^0(\overline{E}, n/2\pi) \leq n/2,$$

already considered in (4.58) and (4.59). (These estimates follow from the definition of \tilde{h}_{Ar}^0 and from (4.52).)

According to Corollary 4.5.8 applied with $t = n/2\pi$, for any $n \in \mathbb{N}_{\geq 2}$ and any $\delta \in \mathbb{R}$, there exists an euclidean lattice of rank n and Arakelov degree δ such that

$$h_{\text{Ar}}^0(\overline{E}_+, n/2\pi) - h_{\theta}^0(\overline{E}_+) \geq \log \frac{1 + v_n(n/2\pi)^{n/2} e^{\delta}}{1 + e^{\delta}}.$$

Besides,

$$\lim_{\delta \rightarrow +\infty} \log \frac{1 + v_n(n/2\pi)^{n/2} e^{\delta}}{1 + e^{\delta}} = \log[v_n(n/2\pi)^{n/2}],$$

and, when n goes to $+\infty$,

$$\log[v_n(n/2\pi)^{n/2}] = n/2 + O(\log n).$$

This shows notably that the “best constant” in the right-hand side of (4.69) is equivalent to $n/2$ when n goes to $+\infty$.

5. COUNTABLY GENERATED PROJECTIVE MODULES AND LINEARLY COMPACT TATE SPACES OVER DEDEKIND RINGS

In this section, we denote by A a Dedekind ring (in the sense of Bourbaki, [Bou65], VII.2.1; in other words, A is either a field, or a Noetherian integrally closed domain of dimension 1) and we introduce some categories of (topological) modules CP_A and CTC_A attached to A . When the Dedekind ring A is the ring \mathcal{O}_K of integers in some number field K , the modules in these categories will occur in the following sections as the \mathcal{O}_K -modules underlying the “infinite dimensional hermitian vector bundles” over $\text{Spec } \mathcal{O}_K$ investigated in this article.

The objects in the dual categories CP_A and CTC_A are easily described. Namely, an object of CP_A is an A -module which is, either finitely generated and projective, or isomorphic to $A^{(\mathbb{N})}$. An object of CTC_A is a topological A -module¹⁹ which is, either a finitely generated projective A -module equipped with the discrete topology, or isomorphic to $A^{\mathbb{N}}$ equipped with the product of the discrete topology on every factor A .

Handling the morphisms in these categories requires more care. The *strict* morphisms in CTC_A play an especially important role, as shown in Section 5.4, and diverse “pathologies” concerning the morphisms in CP_A and CTC_A occur naturally, as demonstrated by the examples in Section 5.5.

5.1. Countably generated projective A -modules.

5.1.1. *The category CP_A .* The following proposition is a simple consequence of the fact that, over a Dedekind ring, a finitely generated module is projective when it is torsion free.

Proposition 5.1.1. *For any A -module M , the following conditions are equivalent:*

- (1) *The A -module M is countably generated and projective.*
- (2) *The A -module M is isomorphic to a direct summand of $A^{(\mathbb{N})}$.*
- (3) *The A -module M is isomorphic to some A -submodule of $A^{(\mathbb{N})}$.*
- (4) *There exists a family $(M_i)_{i \in \mathbb{N}}$ of A -submodules of M such that:*
 - (i) *for any $i \in \mathbb{N}$, M_i is a finitely generated torsion free A -module;*
 - (ii) *for any $i \in \mathbb{N}$, M_i is a saturated A -submodule of M_{i+1} ;*
 - (iii) *$M = \bigcup_{i \in \mathbb{N}} M_i$.*
- (5) *The A -module M is a countable direct sum of finitely generated projective A -modules.*

¹⁹By a *topological A -module*, we mean a topological A -module over the ring A equipped with the discrete topology.

Proof of Proposition 5.1.1. The implications (5) \Rightarrow (1) \Rightarrow (2) \Rightarrow (3) are clear.

When (3) holds, we may consider the filtration $(N_i)_{i \in \mathbb{N}}$ of $N := A^{(\mathbb{N})}$ defined by

$$N_i := \{(a_k)_{k \in \mathbb{N}} \in A^{(\mathbb{N})} \mid \forall k \in \mathbb{N}_{\geq i}, a_k = 0\}.$$

Then the filtration $(M_i)_{i \in \mathbb{N}}$ of M defined by $M_i := M \cap N_i$ satisfies (4).

When (4) holds, for every $i \in \mathbb{N}$, the quotient M_{i+1}/M_i is a finitely generated A -module, which is torsion free, hence projective. Therefore the short exact sequence of A -modules

$$0 \longrightarrow M_i \longrightarrow M_{i+1} \longrightarrow M_{i+1}/M_i \longrightarrow 0$$

is split, and there exists a (necessarily finitely generated and projective) A -submodule P_i of M_{i+1} such that $M_{i+1} = M_i \oplus P_i$. Then we obtain the following decomposition of M :

$$M = M_0 \oplus \bigoplus_{i \in \mathbb{N}} P_i.$$

This displays M as a countable direct sum of finitely generated projective A -modules. \square

We define the A -linear category CP_A of *countably generated projective A -modules* as the category whose objects are A -modules satisfying the equivalent conditions in Proposition 5.1.1, and whose morphisms are A -linear maps.

For any object M of CP_A , we denote by $\mathcal{F}(M)$ the family of finitely generated A -submodules of M , and by $\mathcal{FS}(M)$ the family of saturated finitely generated A -submodules of M . We shall also denote by $\text{co}\mathcal{F}(M)$ (resp. by $\text{co}\mathcal{FS}(M)$) the family of A -submodules M' of M such that M/M' is a finitely generated A -module (resp. a finitely generated torsion free A -module).

Observe that, according to the equivalences of conditions (1) and (3) in Proposition 5.1.1, any A -submodule M' of an object M of CP_A is again an object of CP_A . However, the quotient A -module M/M' — even if assumed torsion-free — is not always an object in CP_A (see for instance, when $A = \mathbb{Z}$, the constructions in Proposition 5.5.1, in paragraph 5.5 *infra*, notably the short exact sequence 5.32).

The additive category admits obvious finite direct sums, that are also finite direct products. It actually admits countable direct sums.

If B denotes a Dedekind ring which is an A -algebra, the tensor product defines an additive functor:

$$\cdot \otimes_A B : CP_A \longrightarrow CP_B.$$

5.1.2. A theorem of Kaplansky. As any finitely generated projective A -module is a (finite) direct sum of invertible A -modules, Condition (5) is equivalent to M being a countable direct sum of invertible A -modules. Actually, when M has infinite rank, this observation admits the following strengthening, proved by Kaplansky ([Kap52], Theorem 2) in a more general setting:

Proposition 5.1.2. *If some A -module M satisfies the conditions in Proposition 5.1.1 and has infinite rank (or equivalently, is not finitely generated), then it is free, hence isomorphic to $A^{(\mathbb{N})}$.*

When A is principal (e.g., when $A = \mathbb{Z}$, a case of special interest in this article), this is straightforward. The part of Kaplansky's argument in *loc. cit.* relevant to the derivation of Proposition 5.1.2 for a general Dedekind ring A may be summarized as follows.

Firstly one shows that, for any element m of some projective countably generated A -module M of finite rank, there exists a direct summand P in M , free and of finite rank, which contains m .

To achieve this, observe that M may be written as an infinite countable direct sum $\bigoplus_{i \in \mathbb{N}} I_i$ of invertible submodules I_i of M , and recall that, for any two invertible A -modules I and J , the A -modules $I \oplus J$ and $A \oplus (I \otimes J)$ are isomorphic. This last fact implies that, for any $n \in \mathbb{N}$, if we

define $J_n := \bigotimes_{0 \leq i \leq n} I_i$, then the A -module $\bigoplus_{0 \leq i \leq n} I_i \oplus J^\vee$ is free of rank $n + 1$, and that there exists some isomorphism of A -modules:

$$\phi : I_{n+1} \oplus I_{n+2} \xrightarrow{\sim} J^\vee \oplus (J \otimes I_{n+1} \otimes I_{n+2}).$$

Therefore, if n is chosen so large that $\bigoplus_{0 \leq i \leq n} I_i$ contains m , then the submodule $P := \bigoplus_{0 \leq i \leq n} I_i \oplus \phi^{-1}(J^\vee \oplus \{0\})$ of M is a free direct summand, of rank $n + 1$, and contains m .

Secondly one considers a countable family of generators $(m_i)_{i \in \mathbb{N}_{>0}}$ of M , and by means of the above fact, one constructs inductively projective A -submodules $(P_i)_{i \in \mathbb{N}_{>0}}$ and $(M^i)_{i \in \mathbb{N}}$ of M , such that the P_i are free of finite rank and the M^i are countably generated, and such that the following conditions are satisfied:

- (1) $M^0 = M$;
- (2) for any $i \in \mathbb{N}_{>0}$, $M^{i-1} = P_i \oplus M^i$ and $m_i \in P_i$.

Thus we obtain a decomposition $M = \bigoplus_{i \in \mathbb{N}_{>0}} P_i$, which shows that M is a countable direct sum of free modules of finites ranks, and completes the proof.

5.2. Linearly compact Tate spaces with countable basis.

5.2.1. Basic definitions. We define the A -linear category CTC_A of *linearly compact Tate spaces with countable basis* over A as follows.

An object N of CTC_A is a topological module over the ring A equipped with the discrete topology which satisfies the following two conditions:

CTC₁ : *The topology of N is Hausdorff and complete.*

CTC₂ : *There exists a countable basis of neighborhoods U of 0 in N consisting in A -submodules of N such that N/U is a finitely generated projective A -module.*

Morphisms in the category CTC_A are A -linear continuous maps: for any two objects N_1 and N_2 in CTC_A , we let

$$\mathrm{Hom}_{CTC_A}(N_1, N_2) := \mathrm{Hom}_A^{\mathrm{cont}}(N_1, N_2).$$

For any subset U of some A -module N , we may consider the condition appearing in **CTC₂**:

C_U : *U is a A -submodule of N , and N/U is a finitely generated projective A -module.*

Then we have:

Lemma 5.2.1. *Let N be an object of CTC_A . A subset U of N is an neighborhood of 0 and satisfies Condition **C_U** if and only if U is an open saturated submodule of N .*

Proof. The necessity is clear. Conversely, if U is an open saturated submodule of N , then it contains a neighborhood U_0 of 0 which satisfies **C_{U₀}**. Then U/U_0 is a saturated submodule of the finitely generated projective A -module N/U_0 , and consequently

$$N/U \simeq (N/U_0)/(U/U_0)$$

also is a finitely generated projective A -module. □

For any object N of CTC_A , we shall denote the family of open saturated submodules of N by $\mathcal{U}(N)$. It is stable under finite intersection.

Any finitely generated projective A -module, equipped with the discrete topology, becomes an object of CTC_A . In this way, the category of finitely generated projective A -modules and A -linear maps appears as a full subcategory of CTC_A .

According to the countability assumption in **CTC₂**, for any object N of CTC_A , there exists a “non-increasing” sequence

$$U_0 \leftarrow U_1 \leftarrow U_2 \leftarrow \dots$$

of submodules in $\mathcal{U}(N)$ which constitute a basis of neighborhoods of 0 in N . We shall call any such sequence $(U_i)_{i \in \mathbb{N}}$ in $\mathcal{U}(\widehat{E})^{\mathbb{N}}$ a *filtration defining the topology of \widehat{E}* , or shortly a *defining filtration* in $\mathcal{U}(\widehat{E})^{\mathbb{N}}$.

From any defining filtration $(U_i)_{i \in \mathbb{N}}$ in $\mathcal{U}(\widehat{E})^{\mathbb{N}}$, we may construct a countable projective system of finitely generated projective A -modules:

$$\widehat{E}/U_0 \longleftarrow \widehat{E}/U_1 \longleftarrow \widehat{E}/U_2 \longleftarrow \dots,$$

and we may consider the canonical morphism $\widehat{E} \longrightarrow \varprojlim_i \widehat{E}/U_i$, defined by the quotient maps $\widehat{E} \longrightarrow \widehat{E}/U_i$. According to Condition **CTC**₁, this morphism is bijective, and actually becomes an isomorphism of topological A -modules

$$(5.1) \quad \widehat{E} \simeq \varprojlim_i \widehat{E}/U_i,$$

when $\varprojlim_i \widehat{E}/U_i$ is equipped with the projective limit topology deduced from the discrete topology on the finitely generated projective modules \widehat{E}/U_i .

Conversely, for any projective system

$$(5.2) \quad E_0 \xleftarrow{q_0} E_1 \xleftarrow{q_1} E_2 \xleftarrow{q_2} \dots$$

of surjective morphisms between finitely generated projective A -modules, the projective limit

$$\widehat{E} := \varprojlim_i E_i,$$

equipped with its natural prodiscrete topology, defines an object of CTC_A .

Moreover, if $\widehat{E} := \varprojlim_i E_i$ and $\widehat{F} := \varprojlim_j F_j$ are two objects of CTC_A , realized as limits of projective systems of finitely generated projective A -modules as above, we have a canonical identification:

$$\mathrm{Hom}_{CTC_A}(\widehat{E}, \widehat{F}) := \mathrm{Hom}_A^{\mathrm{cont}}(\varprojlim_i E_i, \varprojlim_j F_j) \simeq \varprojlim_j \varinjlim_i \mathrm{Hom}_A(E_i, F_j).$$

If B denotes a Dedekind ring which is an A -algebra, the *completed* tensor product defines an additive functor:

$$\cdot \hat{\otimes}_A B : CTC_A \longrightarrow CTC_B.$$

Observe that, for any projective system (5.2) of surjective morphisms of finitely projective A -modules, we get, by extending the scalars from A to B , a projective system of surjective morphisms of finitely projective B -modules

$$(5.3) \quad E_{0B} \xleftarrow{q_{0B}} E_{1B} \xleftarrow{q_{1B}} E_{2B} \xleftarrow{q_{2B}} \dots$$

Its projective limit “is” the object of CTC_B deduced from the projective limit of (5.2) by the completed tensor product functor. Indeed, we have a canonical isomorphism of prodiscrete B -modules:

$$(\varprojlim_i E_i) \hat{\otimes}_A B \xrightarrow{\sim} \varprojlim_i E_{iB}.$$

5.2.2. Subobjects and countable products.

Proposition 5.2.2. *Let N be an object of CTC_A . Any closed A -submodule N' of N , equipped with the induced topology, is an object of CTC_A .*

Proof. Equipped with the induced topology, N' is clearly Hausdorff and complete. Therefore the topological A -module N' satisfies **CTC**₁.

Moreover, if some neighborhood U of 0 in N satisfies **C**_U, then $U' := U \cap N'$ is a neighborhood of 0 in N' which is clearly an A -submodule of N' . Moreover the inclusion $N' \hookrightarrow N$ defines an injective morphism of A -modules $N'/U' \rightarrow N/U$; therefore, N'/U' — as any submodule of a finitely

generated projective module over a Dedekind ring — is also a finitely generated projective A -module. In other words, U' satisfies $\mathbf{C}_{U'}$.

This immediately implies that condition \mathbf{CTC}_2 also is inherited by N' . \square

Observe that, with the notation of Lemma 5.2.2, the topological A -module N/N' , even if assumed torsion-free, may not be an object of \mathbf{CTC}_A .

For instance, when $A = \mathbb{Z}$, the short exact sequences²⁰ (5.35) and (5.43) in Proposition 5.5.1 *infra* and its proof display the ring of p -adic integers \mathbb{Z}_p , equipped with its p -adic topology, as a quotient of $\mathbb{Z}^{\mathbb{N}}$ by a closed submodule. Actually, one may easily show that *the topological \mathbb{Z} -modules that may be realized as quotient of an object of $\mathbf{CTC}_{\mathbb{Z}}$ by a closed subobject are precisely the commutative Polish topological groups G admitting a basis of neighborhoods of 0 which are open subgroups U such that G/U is finitely generated.*

The additive category \mathbf{CTC}_A admits obvious finite direct sums, that are also finite direct products.

It also admits *countable products*. Indeed, if $(N_i)_{i \in I}$ is a countable family of objects in \mathbf{CTC}_A , the A -module

$$N := \prod_{i \in I} N_i,$$

equipped with the product topology, is easily seen to define an object in \mathbf{CTC}_A . The projection maps $\text{pr}_i : N \rightarrow N_i$ are morphisms in \mathbf{CTC}_A , and $(n, (p_i)_{i \in I})$ is a product of the N_i 's in the category \mathbf{CTC}_A .

More generally, any projective system of surjective open morphisms in \mathbf{CTC}_A

$$M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow M_3 \leftarrow \dots$$

admits a (projective) limit $\varprojlim_i M_i$ in \mathbf{CTC}_A , defined by the A -module projective limit of the M_i 's, equipped with the projective limit of their discrete topology.

The following proposition shows that, up to isomorphism, every object of \mathbf{CTC}_A is a product of finitely generated projective A -modules (equipped with the discrete topology). Its easy proof is left to the reader.

Proposition 5.2.3. *Consider a projective system of surjective morphisms of finitely generated projective A -modules:*

$$(5.4) \quad E_0 \xleftarrow{q_0} E_1 \xleftarrow{q_1} E_2 \xleftarrow{q_2} \dots$$

For every $i \in \mathbb{N}$, there exists an A -linear section $\sigma_i : E_i \rightarrow E_{i+1}$ of q_i , and if we define

$$\begin{aligned} S_i &:= \ker q_{i-1} && \text{if } i \geq 1, \\ &:= E_0 && \text{if } i = 0, \end{aligned}$$

then the A -modules S_i are finitely generated and projective, and for every $i \in \mathbb{N}_{>0}$, we have:

$$(5.5) \quad E_i = S_i \oplus \sigma_{i-1}(E_{i-1}).$$

The direct sum decompositions (5.5) determine a family of isomorphisms of A -modules

$$\iota_n : E_n \xrightarrow{\sim} \bigoplus_{0 \leq i \leq n} S_i$$

such that

$$\iota_n \circ q_n \circ \iota_{n+1}^{-1} : \bigoplus_{0 \leq i \leq n+1} S_i \longrightarrow \bigoplus_{0 \leq i \leq n} S_i$$

²⁰Observe that these are actually strict short exact sequences of topological abelian groups.

is the projection map on the first $n + 1$ -th factors, and consequently an isomorphism in CTC_A :

$$\iota : \varprojlim_n E_n \xrightarrow{\sim} \prod_{i \in \mathbb{N}} S_i.$$

5.2.3. Continuity of morphisms of A -modules between objects of CTC_A . In this paragraph, we want to indicate that, for a large class of Dedekind rings A , any morphism of A -modules between two objects N_1 and N_2 in CTC_A is automatically continuous. In other words, for these rings, the forgetful functor from the category CTC_A to the category of A -modules is fully faithful.

This will follow from the variant of results of Specker ([Spe50]) and Enochs ([Eno64]) discussed in Appendix B.

Observe that, for any Dedekind ring A , precisely one of the following three conditions is satisfied:

Ded₁ : A is a field;

Ded₂ : A is a complete discrete valuation ring;

Ded₃ : there exists some non-zero prime ideal \mathfrak{p} of A such that A is not \mathfrak{p} -adically complete.²¹

For instance, any countable Dedekind ring A which is not a field satisfies **Ded₃**, since the cardinality of a complete discrete valuation ring is at least the cardinality of the continuum.

When the Dedekind ring A satisfies **Ded₁** or **Ded₂**, there exists “many elements” in $\text{Hom}_A(A^{\mathbb{N}}, A)$ which are not continuous when $A^{\mathbb{N}}$ (resp. A) is equipped with its natural prodiscrete (resp. discrete) topology, or equivalently, there exists some A -linear map $\xi : A^{\mathbb{N}} \rightarrow A$ which is not of the form

$$(5.6) \quad \xi((x_n)_{n \in \mathbb{N}}) = \sum_{n \in \mathbb{N}} \xi_n x_n$$

for some $(\xi_n)_{n \in \mathbb{N}}$ in $A^{(\mathbb{N})}$.

Indeed, when A is a field, any non-zero linear form on the vector space $A^{\mathbb{N}}$ that vanishes on its subspace $A^{(\mathbb{N})}$ is such a map. When A is a complete discrete valuation ring, of maximal ideal \mathfrak{m} , for any sequence $(\xi_n)_{n \in \mathbb{N}}$ in $A^{\mathbb{N}} \setminus A^{(\mathbb{N})}$ that converges to zero in the \mathfrak{m} -adic topology, the formula (5.6) defines an element $\xi \in \text{Hom}_A(A^{\mathbb{N}}, A)$ of the required type.

In contrast, for Dedekind rings satisfying **Ded₃**, the following continuity results holds:

Proposition 5.2.4. *If the Dedekind ring A satisfies **Ded₃**, then for any two objects N and N' in CTC_A , we have:*

$$(5.7) \quad \text{Hom}_A(N, N') = \text{Hom}_A^{\text{cont}}(N, N').$$

Proof. When $N' = A$, this follows from Corollary B.2.2 applied to $R := A$ and to $M := N$. (Indeed, we may choose as \mathfrak{m} any non-zero prime ideal \mathfrak{p} of A : the local ring $A_{(\mathfrak{p})}$ is not complete, since A satisfies **Ded₃**.)

The validity of (5.7) when $N' = A$ implies its validity when $N' = A^{\oplus n}$ for some $n \in \mathbb{N}$, hence its validity for any finitely generated projective A -module N' equipped with the discrete topology. (Indeed an such module may be realized as a submodule — actually as a direct summand — of some free A -module $A^{\oplus n}$.)

To complete the proof of Proposition 5.2.4, observe that any object N' of CTC_A may be realized as the projective limit $N' = \varprojlim_i N'_i$ of some projective system of finitely generated projective A -modules $N'_0 \leftarrow N'_1 \leftarrow N'_2 \leftarrow \dots$, and that we have natural identifications:

$$\text{Hom}_A(N, N') \simeq \varprojlim_i \text{Hom}_A(N, N'_i)$$

²¹It is straightforward that, when **Ded₃** holds, the ring A is not \mathfrak{p} -adically complete for *every* non-zero prime ideal \mathfrak{p} of A .

and

$$\mathrm{Hom}_A^{\mathrm{cont}}(N, N') \simeq \varprojlim_i \mathrm{Hom}_A^{\mathrm{cont}}(N, N'_i).$$

The validity of (5.7) consequently follows from the already established equalities:

$$\mathrm{Hom}_A(N, N'_i) = \mathrm{Hom}_A^{\mathrm{cont}}(N, N'_i).$$

□

5.2.4. The topology on objects in CTC_A when A is a topological ring. Let us assume that, besides its discrete topology, the ring A is equipped with a topology that makes A a topological ring. We shall denote A^{an} the topological ring defined by A equipped with this finer topology.

For instance, A may be a discrete valuation ring and A^{an} the ring A equipped with the topology associated to its discrete valuation, or a local field and A^{an} the field A equipped with its “usual” locally compact topology.

In this situation, besides its topology of pro-discrete A -module, any object N of CTC_A is canonically endowed with a finer topology, which makes it a topological A^{an} -module N^{an} .

Indeed, any finitely generated projective A -module P is equipped with a canonical topology of A^{an} -module: if P is embedded as a direct summand in the direct sum $A^{\oplus n}$ of a finite number of copies of A , this topology is the one induced by the product topology on $(A^{\mathrm{an}})^n$. We shall denote by P^{an} the so-defined topological A^{an} -module. Observe that any A -linear morphism $\phi : P_1 \rightarrow P_2$ between finitely generated projective A -modules defines a continuous morphism $\phi : P_1^{\mathrm{an}} \rightarrow P_2^{\mathrm{an}}$ of topological A^{an} -modules.

By definition, if N denotes an object of CTC_A , we have a canonical isomorphism of topological A -modules

$$(5.8) \quad N \xrightarrow{\sim} \varprojlim_{U \in \mathcal{U}(N)} N/U,$$

where $\varprojlim_{U \in \mathcal{U}(N)} N/U$ is equipped with the pro-discrete topology. We make N a topological A^{an} -module N^{an} by declaring (5.8) to be an isomorphism of topological A^{an} -modules

$$(5.9) \quad N^{\mathrm{an}} \xrightarrow{\sim} \varprojlim_{U \in \mathcal{U}(N)} (N/U)^{\mathrm{an}},$$

where $\varprojlim_{U \in \mathcal{U}(N)} (N/U)^{\mathrm{an}}$ is equipped with the projective limit topology deduced from the canonical topology of A^{an} -module on each finitely generated projective A -module N/U .

Any $U \in \mathcal{U}(N)$ is an object of CTC_A and as such is equipped with the “analytic” topology U^{an} . One easily see that U is actually closed in N^{an} and that the topology of U^{an} is the topology induced by the topology of N^{an} .

Any morphism $f : N' \rightarrow N$ in CTC_A defines a continuous morphism of topological A^{an} -modules $f : N'^{\mathrm{an}} \rightarrow N^{\mathrm{an}}$. However the so defined injective map

$$\mathrm{Hom}_A^{\mathrm{cont}}(N', N) \hookrightarrow \mathrm{Hom}_{A^{\mathrm{an}}}^{\mathrm{cont}}(N'^{\mathrm{an}}, N^{\mathrm{an}})$$

is not surjective in general.

When A^{an} is the field \mathbb{R} (resp. \mathbb{C}) equipped with its usual “analytic” topology, the topological A^{an} -module N^{an} is a topological vector space over \mathbb{R} (resp. over \mathbb{C}), isomorphic to \mathbb{R}^n (resp. to \mathbb{C}^n) if $n := \dim_{A^{\mathrm{an}}} N$ is finite, and to $\mathbb{R}^{\mathbb{N}}$ (resp. to $\mathbb{C}^{\mathbb{N}}$) if n is infinite. In particular, it is a Fréchet locally convex vector space over \mathbb{R} (resp. over \mathbb{C}).

5.3. The duality between CP_A and CTC_A . In this section, we discuss the (anti)equivalence of A -linear categories between CP_A and CTC_A defined by duality, and some of its consequences.

5.3.1. The duality functors.

(i) To any A -module M , we may attach its dual topological A -module, namely the A -module

$$(5.10) \quad M^\vee := \text{Hom}_A(M, A)$$

equipped with the topology of pointwise convergence. If $\alpha : M_2 \rightarrow M_1$ is a morphism of A -modules, its transpose

$$(5.11) \quad \alpha^\vee := . \circ \alpha : M_1^\vee \rightarrow M_2^\vee$$

is clearly A -linear and continuous.

For any family $(M_i)_{i \in I}$ of A -modules, there is a canonical identification of topological A -modules:

$$(\bigoplus_{i \in I} M_i)^\vee \xrightarrow{\sim} \prod_{i \in I} M_i^\vee.$$

Besides, if M is a finitely generated projective A -module, then its dual M^\vee also is finitely generated and projective, and the topology of M^\vee is discrete. (Indeed, this holds for any finitely generated free A -module $A^{\oplus n}$, and consequently for any direct summand of such an A -module.)

As any object in CP_A is a countable direct sum of finitely generated projective A -modules, these observations show that the constructions (5.10) and (5.11) define a contravariant A -linear duality functor:

$$(5.12) \quad .^\vee : CP_A \rightarrow CTC_A.$$

(ii) Conversely, to any topological A -module N , we may attach its topological dual, namely the A -module

$$(5.13) \quad N^\vee := \text{Hom}_A^{\text{cont}}(N, A)$$

consisting of *continuous* A -linear maps from N to A . Then any continuous A -linear morphism $\beta : N_1 \rightarrow N_2$ of topological A -modules defines, by transposition, a morphism of A -modules:

$$(5.14) \quad \beta^\vee := . \circ \beta : N_2^\vee \rightarrow N_1^\vee.$$

For any projective system of topological A -modules

$$(5.15) \quad N_0 \xleftarrow{p_0} N_1 \xleftarrow{p_1} N_2 \xleftarrow{p_2} \dots,$$

we may form the projective limit $\varprojlim_i N_i$, as a topological A -module, and we may also consider the dual inductive system of A -modules:

$$N_0^\vee \xrightarrow{p_0^\vee} N_1^\vee \xrightarrow{p_1^\vee} N_2^\vee \xrightarrow{p_2^\vee} \dots$$

Then there is a canonical identification of A -modules:

$$(\varprojlim_i N_i)^\vee \simeq \varinjlim_i N_i^\vee.$$

Besides, if N is a finitely generated projective A -module, equipped with the discrete topology, then

$$N^\vee = \text{Hom}_A^{\text{cont}}(N, A) = \text{Hom}_A(N, A)$$

is also a finitely generated projective A -module. Moreover, if $q : N' \rightarrow N$ is a surjective morphism of finitely generated projective A -modules, then $p^\vee : N^\vee \rightarrow N'^\vee$ is injective and its image is a direct summand in N'^\vee .

As any object in CTC_A is (up to isomorphism) the projective limit of some countable projective system $E_0 \leftarrow E_1 \leftarrow E_2 \leftarrow \dots$ of surjective morphisms between finitely generated projective A -modules, these observations show that the constructions (5.13) and (5.14) define a contravariant A -linear duality functor:

$$(5.16) \quad .^\vee : CTC_A \rightarrow CP_A.$$

5.3.2. *Duality as an adjoint equivalence.* Observe that, for any object M of CP_A and any object N of CTC_A , the A -modules $\text{Hom}_A^{\text{cont}}(M, N^\vee)$ and $\text{Hom}_A(N, M^\vee)$ may both be identified with the module of A -bilinear maps $M \times N \rightarrow A$ which are continuous in the first variable. In this way, we obtain a systems of bijections

$$(5.17) \quad \text{Hom}_{CTC_A}(M, N^\vee) := \text{Hom}_A^{\text{cont}}(M, N^\vee) \simeq \text{Hom}_A(N, M^\vee) = \text{Hom}_{CP_A^{\text{op}}}(M^\vee, N).$$

Proposition 5.3.1. *The bijections (5.17) defines an adjunction of functors:*

$$.^\vee : CTC_A \rightleftarrows CP_A^{\text{op}} : .^\vee$$

It is actually an adjoint equivalence, whose unit and counit are the natural isomorphism $\eta : I_{CTC_A} \simeq .^{\vee\vee}$ and $\epsilon : .^{\vee\vee} \simeq I_{CP_A^{\text{op}}}$ defined by the biduality isomorphisms

$$\begin{aligned} \epsilon_M : M &\xrightarrow{\sim} \text{Hom}_A^{\text{cont}}(\text{Hom}_A(M, A), A) \\ m &\longmapsto (\xi \longmapsto \xi(m)) \end{aligned}$$

and

$$\begin{aligned} \eta_N : N &\xrightarrow{\sim} \text{Hom}_A(\text{Hom}_A^{\text{cont}}(N, A), A) \\ n &\longmapsto (\zeta \longmapsto \zeta(n)) \end{aligned}$$

associated to any object M of CP_A and to any object N of CTC_A .

Proof. We only sketch the proof and leave the details to the readers.

The naturality with respect to M (in CTC_A) and to N (in CP_A^{op}) of the bijections (5.17) is straightforward, and the expressions for ϵ_M and η_N as well.

To complete the proof, we are left to establish that, for any M (resp. N) in CP_A (resp. in CTC_A), ϵ_M (resp. η_N) is an isomorphism in CP_A (resp. in CTC_A). The compatibility of the duality functors with countable direct sums (in CP_A) and countable products (in CTC_A), and the fact that any object in CP_A (resp. in CTC_A) is a countable direct sum (resp. a countable product) of finitely generated projective A -modules, allows us to reduce the proof to the special case when M (resp. N) is a finitely generated projective A -module. Then it is straightforward. \square

Corollary 5.3.2. *The duality functors (5.12) and (5.16) are (anti)equivalences of categories.* \square

Let B be a Dedekind ring which is an A -algebra. The reader may easily establish the compatibility between the duality functors between CP_A and CTC_A and between CP_B and CTC_B , and of the “base change” functors $. \otimes_A B : CP_A \rightarrow CP_B$ and $. \hat{\otimes}_A B : CTC_A \rightarrow CTC_B$.

5.3.3. *Applications.* Diverse results about the category CP_A may be transferred by duality to the category CTC_A .

(i) For instance, the morphisms ϕ in $\text{Hom}_A(A^{(\mathbb{N})}, A^{(\mathbb{N})})$ are in bijection with “infinite matrices” $(\phi_{ij})_{(i,j) \in \mathbb{N} \times \mathbb{N}}$ in $A^{\mathbb{N} \times \mathbb{N}}$ which admit only a finite of non-zero entries in each column, by the usual formula, valid for any $x = (x_i)_{i \in \mathbb{N}}$ and $y = (y_j)_{j \in \mathbb{N}}$ in $A^{(\mathbb{N})}$:

$$(5.18) \quad y = \phi(x) \Leftrightarrow y_i = \sum_{j \in \mathbb{N}} \phi_{ij} x_j.$$

By duality, this implies that the morphisms ϕ in $\text{Hom}_A^{\text{cont}}(A^{\mathbb{N}}, A^{\mathbb{N}})$ are bijection with matrices $(\phi_{ij})_{(i,j) \in \mathbb{N} \times \mathbb{N}}$ in $A^{\mathbb{N} \times \mathbb{N}}$ which contains only a finite of non-zero entries in each row, still by means of formula (5.18).

(ii) From Kaplansky’s result concerning the freeness of modules of infinite rank in CP_A (Proposition 5.1.2), we get the following refinement of the description of objects of CP_A as countable products of finitely generated projective A -modules in Proposition 5.2.3:

Proposition 5.3.3. *Let N be an object in CTC_A . Either the topology of N is discrete and the A -module N is finitely generated and projective, or N is isomorphic to $A^{\mathbb{N}}$ as a topological A -module.* \square

(iii) When A is a field k , the objects of CTC_A are precisely the *linearly compact k -vector spaces*, introduced by Lefschetz and Chevalley ([Lef42], Chapter II, §6), that admit a countable basis of neighborhoods of zero. In this case, the category CP_A is the category of k -vector spaces of countable dimension. Diverse results from linear algebra over the field k may be transported, by duality, to results concerning CTC_A .

In this way, from the basic facts about k -vector spaces of countable dimension and their k -linear maps, we derive the following results (which go back to Toeplitz and Köthe; see notably [Toe09] and [Köt49]; see also [Die50]):

Proposition 5.3.4. *Let k be a field.*

1) *In the category CTC_k , any object is isomorphic, either to k^n for some non-negative integer n , or to $k^{\mathbb{N}}$.*

2) *For any morphism $\phi : N_1 \rightarrow N_2$ in CTC_k , there exists objects K , N , and C and isomorphisms*

$$u : N_1 \xrightarrow{\sim} K \oplus N \text{ and } v : N_2 \xrightarrow{\sim} C \oplus N$$

in CTC_k such that the morphism $\tilde{\phi} := v\phi u^{-1} : K \oplus N \rightarrow C \oplus N$ is the “bloc diagonal” morphism $0 \oplus Id_N$, which sends $(k, n) \in K \oplus N$ to $(0, n)$ in $C \oplus N$. \square

5.4. Strict morphisms, exactness and duality.

5.4.1. Strict morphisms in CTC_A .

We shall say that a morphism

$$(5.19) \quad \phi : N_1 \rightarrow N_2$$

in CTC_A is *strict* if ϕ is a strict morphism of topological groups, namely if the induced map

$$\begin{array}{ccc} \tilde{\phi} : N_1 / \ker \phi & \longrightarrow & \text{im } \phi \\ [x] & \longmapsto & \phi(x) \end{array}$$

is a homeomorphism, when $N_1 / \ker \phi$ (resp. $\text{im } \phi$) is equipped with the topology quotient of the topology of N_1 (resp., with the topology induced by the topology of N_2).

As N_1 and therefore its quotient $N_1 / \ker \phi$ are complete, if the morphism (5.19) is strict, its image $\text{im } \phi$ also is complete, hence closed in N_2 , and therefore defines an object of CTC_A by Proposition 5.2.2.

Accordingly, the strict morphisms in $\text{Hom}_A^{\text{cont}}(N_1, N_2)$ are precisely the maps $\phi : N_1 \rightarrow N_2$ for which there exists an object I in CTC_A , defined by some closed A -submodule of N_2 , and an *open surjective* morphism ϕ_0 in $\text{Hom}_A^{\text{cont}}(N_1, I)$ such that ϕ admits the factorization

$$\phi : N_1 \xrightarrow{\phi_0} I \hookrightarrow N_2.$$

5.4.2. *Finite rank morphisms.* We want to show that any morphism in CTC_A whose image is finitely generated is strict. This will follow from the following result, of independent interest:

Proposition 5.4.1. *Let P be a finitely generated projective A -module, and let N be an object of CTC_A . Let K denote the field of fractions of A .*

For any morphism of A -modules $\phi : P \rightarrow N$, the following conditions are equivalent:

- (1) *ϕ is injective;*
- (2) *the K -linear map $\phi_K : P_K := P \otimes_A K \rightarrow N_K := N \hat{\otimes}_A K$ is injective;*

(3) there exists U in $\mathcal{U}(N)$ such that the composite morphism of A -modules

$$P \xrightarrow{\phi} M \twoheadrightarrow N/U$$

is injective.

Proof. The implications (3) \Rightarrow (2) \Rightarrow (1) are straightforward.

To prove the implication (1) \Rightarrow (3), observe that the inverse images $\phi^{-1}(V)$ of the submodules V of N in $\mathcal{U}(N)$ constitute a family of saturated submodules of E stable under finite intersection. As P has finite rank, we may consider U in $\mathcal{U}(N)$ such that $\phi^{-1}(U)$ has minimal rank. Then, for any $V \in \mathcal{U}(N)$, the intersection $\phi^{-1}(U) \cap \phi^{-1}(V)$ is saturated in N , of rank at most the rank of $\phi^{-1}(U)$, and therefore coincides with $\phi^{-1}(U)$. This shows that

$$(5.20) \quad \phi^{-1}(U) = \bigcap_{V \in \mathcal{U}(N)} \phi^{-1}(V).$$

When (1) is satisfied,

$$\bigcap_{V \in \mathcal{U}(N)} \phi^{-1}(V) = \phi^{-1}\left(\bigcap_{V \in \mathcal{U}(N)} V\right) = \phi^{-1}(0) = \{0\},$$

and therefore $\phi^{-1}(U) = \{0\}$, and (3) is satisfied. \square

Corollary 5.4.2. *If N is an object of CTC_A and if P is a finitely generated A -submodule of N , then P is a finitely generated projective A -module and the topology of N induces the discrete topology on P .*

Moreover the saturation

$$\tilde{P} := \{n \in N \mid \exists \alpha \in A \setminus \{0\}, \alpha n \in P\}$$

of P in N is also a finitely generated A -module.

Proof. The projectivity of P is clear. Moreover, according to Proposition 5.4.1, we may choose an element U of $\mathcal{U}(N)$ such that $U \cap P = 0$. Since U is open in N , this shows that the topology on P induced by the one of M is the discrete topology.

To prove that \tilde{P} is finitely generated observe that the composite map

$$U \hookrightarrow N \twoheadrightarrow N/P$$

is injective and fits into an exact sequence of A -modules:

$$0 \longrightarrow U \longrightarrow N/P \xrightarrow{q} N/(U+P) \longrightarrow 0.$$

As U is torsion free, the map q defines by restriction an injection of torsion submodules:

$$q : (N/P)_{\text{tor}} \hookrightarrow (N/(U+P))_{\text{tor}}.$$

As N/U — hence $N/(U+P)$ — is a finitely generated A -module, this proves that $(N/P)_{\text{tor}}$ also is finitely generated. Finally the short exact sequence

$$0 \longrightarrow P \longrightarrow \tilde{P} \longrightarrow (N/P)_{\text{tor}} \longrightarrow 0$$

shows that \tilde{P} is finitely generated. \square

Corollary 5.4.3. *If the image of some morphism in CTC_A is a finitely generated A -module, then this morphism is strict.* \square

The following proposition completes Proposition 5.4.1 in the situation when ϕ as a saturated image.

Proposition 5.4.4. *Let P be a finitely generated projective A -module, and let N be an object of CTC_A .*

For any morphism of A -modules $\phi : P \longrightarrow N$, the following conditions are equivalent:

- (1) the morphism ϕ is injective and its image $\phi(P)$ is a saturated A -submodule of N .*
- (2) there exists U in $\mathcal{U}(N)$ such that the composite morphism of A -modules*

$$P \xrightarrow{\phi} M \rightarrow N/U$$

is injective and its image is a saturated A -submodule of N/U .

When condition (2) is satisfied by some U in $\mathcal{U}(N)$, it is clearly also satisfied by any element U' in $\mathcal{U}(N)$ contained in U .

Proof. The implication (2) \Rightarrow (1) is straightforward.

To establish the converse implication, let us assume that ϕ is injective with saturated image, and consider a defining sequence $(U_i)_{i \in \mathbb{N}}$ in $\mathcal{U}(N)$. We define

$$N_i := M/U_i$$

and we denote by

$$p_i : N \longrightarrow N_i$$

the canonical quotient map and by

$$\phi_i := p_i \circ \phi : P \longrightarrow N_i$$

its composition with ϕ .

Let \mathfrak{p} be a non-zero prime ideal of A , and let $\mathbb{F}_{\mathfrak{p}} := A/\mathfrak{p}$ be its residue field.

The $\mathbb{F}_{\mathfrak{p}}$ -vector space $N_{\mathbb{F}_{\mathfrak{p}}} := N/\mathfrak{p}N$ may be identified with the tensor product $N \otimes_A \mathbb{F}_{\mathfrak{p}}$. Equipped with the topology quotient of the topology of N , it coincides with the object of $CTC_{\mathbb{F}_{\mathfrak{p}}}$ defined as the completed tensor product $N \hat{\otimes}_A \mathbb{F}_{\mathfrak{p}}$, and also with the projective limit $\varprojlim_i N_{i, \mathbb{F}_{\mathfrak{p}}}$ of the finite dimensional $\mathbb{F}_{\mathfrak{p}}$ -vector spaces $N_{i, \mathbb{F}_{\mathfrak{p}}}$ equipped with the discrete topology.

The $\mathbb{F}_{\mathfrak{p}}$ -linear map

$$\phi_{\mathbb{F}_{\mathfrak{p}}} : P_{\mathbb{F}_{\mathfrak{p}}} \longrightarrow N_{\mathbb{F}_{\mathfrak{p}}}$$

is injective, since ϕ is injective and its image is saturated in N . The map $\phi_{\mathbb{F}_{\mathfrak{p}}}$ is also defined by the compatible system of $\mathbb{F}_{\mathfrak{p}}$ -linear maps

$$\phi_{i, \mathbb{F}_{\mathfrak{p}}} : P_{\mathbb{F}_{\mathfrak{p}}} \longrightarrow N_{i, \mathbb{F}_{\mathfrak{p}}}.$$

Therefore

$$\bigcap_{i \in \mathbb{N}} \ker \phi_{i, \mathbb{F}_{\mathfrak{p}}} = \ker \phi_i = \{0\}.$$

This shows the existence of some $\iota(\mathfrak{p})$ in \mathbb{N} such that $\ker \phi_{i, \mathbb{F}_{\mathfrak{p}}} = \{0\}$ for every $i \in \mathbb{N}_{\geq \iota(\mathfrak{p})}$.

Besides, according to Proposition 5.4.1, there exists ι_0 in \mathbb{N} such that ϕ_{ι_0} (and consequently ϕ_i for any $i \in \mathbb{N}_{\geq \iota_0}$) is injective. The set \mathcal{T} of non-zero prime ideals in A such that

$$\phi_{\iota_0, \mathbb{F}_{\mathfrak{p}}} : P_{\mathbb{F}_{\mathfrak{p}}} \longrightarrow N_{\iota_0, \mathbb{F}_{\mathfrak{p}}}$$

is non-injective is finite. It is indeed defined by the vanishing of the non-zero section $\wedge^{\text{rk } P} \phi_{\iota_0}$ of $(\wedge^{\text{rk } P} P)^{\vee} \otimes \wedge^{\text{rk } P} N_{\iota_0}$ over $\text{Spec } A$.

Finally, if we let

$$\iota_1 := \max(\iota_0, \max_{\mathfrak{p} \in \mathcal{T}} \iota(\mathfrak{p})),$$

then, for any $i \in \mathbb{N}_{\geq \iota_1}$ and any non-zero prime ideal \mathfrak{p} of A , $\phi_{i, \mathbb{F}_{\mathfrak{p}}}$ is injective, and therefore ϕ_i is injective with a saturated image. \square

5.4.3. *Strict short exact sequences and duality.* We shall say that a diagram

$$(5.21) \quad 0 \longrightarrow N_1 \xrightarrow{f} N_2 \xrightarrow{g} N_3 \longrightarrow 0$$

of object and morphisms in CTC_A is a *strict short exact sequence* if it is a short exact sequence of A -modules and if the morphisms f and g are strict — in other words, if f establishes an isomorphism of topological A -modules from N_1 onto $\ker g$ and if g is a surjective open map.

This last condition on g is satisfied notably if g admits a section h (that is, a right inverse) in $\text{Hom}_A^{\text{cont}}(N_3, N_1)$. When this holds, the strict short exact sequence (5.21) is said to be *split*, and h is called a *splitting* of (5.21).

We shall say that a closed A -submodule N' of some object N in CTC_A is *supplemented in CTC_A* when there exists a closed A -submodule N'' of N such that the sum map

$$\begin{aligned} N' \oplus N'' &\longrightarrow N \\ (n', n'') &\longmapsto n' + n'' \end{aligned}$$

is an isomorphism of topological A -modules (in CTC_A , by Proposition 5.2.2).

Observe that the strict short exact sequence (5.21) is split precisely when the closed submodule $f(N_1)$ of N_2 is supplemented in CTC_A . Indeed the splittings h in of (5.21) are in bijections with the “topological supplements” N'' in CTC_A of $f(N_1)$ by the map which sends h to its image $h(N_3)$.

Let M_1, M_2 , and M_3 be objects in CP_A , and let $N_1 := M_1^\vee, N_2 := M_2^\vee$, and $N_3 := M_3^\vee$ be the dual objects in CTC_A .

Let \mathcal{S} (resp. \mathcal{T}) be the subset of $\text{Hom}_A(M_3, M_2) \times \text{Hom}_A(M_2, M_1)$ (resp. of $\text{Hom}_A^{\text{cont}}(N_1, N_2) \times \text{Hom}_A^{\text{cont}}(N_2, N_3)$) consisting of the pairs of morphisms (i, p) (resp. (j, q)) such that the diagram

$$(5.22) \quad 0 \longrightarrow M_3 \xrightarrow{i} M_2 \xrightarrow{p} M_1 \longrightarrow 0$$

is an exact sequence of A -modules (resp. such that

$$(5.23) \quad 0 \longrightarrow N_1 \xrightarrow{j} N_2 \xrightarrow{q} N_3 \longrightarrow 0$$

is a strict short exact sequence in CTC_A).

For each $i \in \{1, 2, 3\}$, the topological dual N_i^\vee of N_i will be identified with M_i by the biduality isomorphisms $\epsilon_{M_i}^{-1}$ of Proposition 5.3.1.

Proposition 5.4.5. *With the above notation, one defines a bijection $\delta : \mathcal{S} \xrightarrow{\sim} \mathcal{T}$ by the formula:*

$$(5.24) \quad \delta(i, p) := (p^\vee, i^\vee).$$

The inverse bijection δ^{-1} is given by

$$(5.25) \quad \delta^{-1}(j, q) = (q^\vee, j^\vee).$$

Moreover, for any (i, p) in \mathcal{S} (resp., for any (j, q) in \mathcal{T}), the short exact sequence of A -module (5.22) (resp. the strict short exact sequence in CTC_A (5.23)) is split.

Proof. 1) For any (i, p) in \mathcal{S} , the associated short exact sequence of A -modules (5.22) is split, since M_1 is projective. This implies that the diagram

$$0 \longrightarrow M_1^\vee \xrightarrow{p^\vee} M_2^\vee \xrightarrow{i^\vee} M_3^\vee \longrightarrow 0$$

deduced from (5.22) by duality is a split strict short exact sequence in CTC_A .

This shows in particular that (p^\vee, i^\vee) belongs to \mathcal{T} .

2) Consider an element (j, q) of \mathcal{T} .

If we apply the functor $\text{Hom}_A^{\text{cont}}(\cdot, A)$ to the strict short exact sequence (5.23), we get the following exact sequence of A -modules:

$$0 \longrightarrow N_3^\vee \xrightarrow{q^\vee} N_2^\vee \xrightarrow{j^\vee} N_1^\vee.$$

Indeed the injectivity of q^\vee follows from the surjectivity of Q , and the equality $\text{im } q^\vee = \ker j^\vee$ means that the continuous A -linear maps from N_2 to A which vanishes on $j(N_1)$ are in bijection — by means of factorization through q — which the A -linear forms from N_3 to A : this follows from the fact that q is continuous and open, of kernel $j(N_1)$.

The morphism $j^\vee : N_2^\vee \longrightarrow N_1^\vee$ may be factorized as

$$(5.26) \quad j^\vee = \iota \circ [j^\vee] : N_2^\vee \xrightarrow{[j^\vee]} \text{im } j^\vee \xhookrightarrow{\iota} N_1^\vee,$$

where ι denotes the inclusion morphism. Moreover the A -module $\text{im } j^\vee$, as any submodule of N_1^\vee , is an object of CP_A .

Finally the diagram

$$(5.27) \quad 0 \longrightarrow N_3^\vee \xrightarrow{q^\vee} N_2^\vee \xrightarrow{[j^\vee]} \text{im } j^\vee \longrightarrow 0$$

is a short exact sequence in CP_A .

According to part 1) of the proof, the short exact sequence (5.27) is split and determines by duality a split short exact sequence in CTC_A . Moreover the factorization (5.26) shows that $j = [j^\vee]^\vee \circ \iota^\vee$. Therefore the following diagram in CTC_A is commutative, and its lines are short strict exact sequences:

$$(5.28) \quad \begin{array}{ccccccc} 0 & \longrightarrow & (\text{im } j^\vee)^\vee & \xrightarrow{[j^\vee]^\vee} & N_2 & \xrightarrow{q} & N_3 \longrightarrow 0 \\ & & \uparrow \iota^\vee & & \parallel & & \parallel \\ 0 & \longrightarrow & N_1 & \xrightarrow{j} & N_2 & \xrightarrow{q} & N_3 \longrightarrow 0. \end{array}$$

This implies that ι^\vee is an isomorphism in CTC_A . In particular, the second line in (5.28), like the first one, is a split short exact sequence in CTC_A .

Moreover, since the duality functor ${}^\vee$ is an equivalence of category from CP_A to CTC_A , this also proves that ι is an isomorphism in CP_A . This establishes the equality $\text{im } j^\vee = N_1^\vee$ and the exactness of

$$0 \longrightarrow N_3^\vee \xrightarrow{q^\vee} N_2^\vee \xrightarrow{j^\vee} N_1^\vee \longrightarrow 0.$$

This shows that (q^\vee, j^\vee) belongs to \mathcal{S} .

The fact that the maps between \mathcal{S} and \mathcal{T} defined by (5.24) and 5.25) are inverse to each other is a straightforward consequence of the “biduality” established in Proposition 5.3.1. \square

For later reference, we spell out some consequences of the results on short exact sequences in CP_A and CTC_A established in the previous proposition:

Proposition 5.4.6. *Let M and M' be two objects in CP_A and let $N := M^\vee$ and $N' := M'^\vee$ be their duals in CTC_A .*

Let $\alpha : M \longrightarrow M'$ be a morphism of A -modules and let $\beta := \alpha^\vee : N' \longrightarrow N$ denote the dual morphism, in $\text{Hom}_A^{\text{cont}}(N', N)$.

1) *The following two conditions are equivalent:*

E₁ : *The morphism β is surjective and strict.*

E₂ : *The morphism α is injective and its cokernel is a projective A -module.*

When these conditions are realized, $\text{im } \alpha$ is a direct summand of M' and $\ker \beta$ is a closed submodule of N supplemented in CTC_A .

2) *The following two conditions are equivalent:*

F₁ : *The morphism β is injective and strict, and the topological A -module $N/\text{im } \beta$ is an object of CTC_A .*

F₂ : *The morphism α is surjective.*

When these conditions are realized, $\ker \alpha$ is a direct summand of M and $\operatorname{im} \beta$ is a closed submodule of N supplemented in CTC_A . \square

Corollary 5.4.7. *A closed A -submodule N' of some object N in CTC_A is supplemented in CTC_A if and only if the quotient topological A -module N/N' is an object of CTC_A . \square*

Corollary 5.4.8. *Let M be an object of CP_A and let $N := M^\vee$ be the dual object in CTC_A .*

Any submodule of M (resp. of N) in $\mathcal{FS}(M)$ (resp. in $\mathcal{U}(N)$) is supplemented in M (resp. in N).

Moreover there is an inclusion reversing bijection

$$.\perp : \mathcal{FS}(M) \xrightarrow{\sim} \mathcal{U}(N).$$

It sends a module M' in $\mathcal{FS}(M)$ to

$$M'^\perp := \{\xi \in N \mid \forall m \in M', \xi(m) = 0\}.$$

The inverse bijection sends an element U in $\mathcal{U}(N)$ to

$$U^\perp := \{m \in M \mid \forall \xi \in U, \xi(m) = 0\}.$$

Moreover, when $U = M'^\perp$, there exists a unique isomorphism of A -modules $I : N/U \xrightarrow{\sim} M'^\vee$ such that, if we denote by $i_{M'} : M' \hookrightarrow M$ the inclusion morphism, the following diagram is commutative:

$$(5.29) \quad \begin{array}{ccc} N & \xrightarrow{=} & M^\vee \\ p_U \downarrow & & \downarrow i_{M'}^\vee \\ N/U & \xrightarrow{\sim} & M'^\vee. \end{array}$$

\square

5.4.4. Strict morphisms and Conditions $\mathbf{Ded}_1, \mathbf{Ded}_2, \mathbf{Ded}_3$. The significance of being strict for a morphism in CTC_A turns out to depend on which of the conditions \mathbf{Ded}_1 , \mathbf{Ded}_2 , or \mathbf{Ded}_3 the base ring A satisfies.

In this paragraph, we present diverse results that illustrate this point. We will return on constructions of non-strict morphisms in the next section, devoted to examples.

Proposition 5.4.9. *If the Dedekind ring A satisfies \mathbf{Ded}_1 — that is, if A is a field — then any morphism in CTC_A is strict.*

Proof. This follows from the description of morphisms in CTC_k in Proposition 5.3.4. Indeed, with the notation of this Proposition, $\tilde{\phi} = 0 \oplus Id_N$ is clearly a strict morphism (of kernel $K \oplus \{0\}$ and of image $\{0\} \oplus N$) and consequently $\phi = v^{-1}\tilde{\phi}u$ is also strict. \square

In Paragraph 5.5.2 *infra*, we shall see that if A satisfies \mathbf{Ded}_2 — that is, if A is a complete discrete valuation ring — then there exists bijective morphisms in CTC_A that are not strict.

Proposition 5.4.10. *When the Dedekind ring A satisfies \mathbf{Ded}_3 , a morphism $\phi : N_1 \rightarrow N_2$ in CTC_A is strict if and only if its image $\operatorname{im} \phi$ is closed in N_2 .*

Moreover, for any two objects N_1 and N_2 of CTC_A , an A -linear map $\phi : N_1 \rightarrow N_2$ is continuous (hence defines a morphism in CTC_A) if and only if its graph is closed in $N_1 \oplus N_2$.

Proof. 1) We have already observed that the image of any strict morphism is closed (for an arbitrary Dedekind ring A).

Conversely, if a morphism $\phi : N_1 \rightarrow N_2$ in CTC_A has its image $\operatorname{im} \phi$ closed in N_2 , then we may form the following diagram in CTC_A , which is an exact sequence of A -modules:

$$(5.30) \quad 0 \rightarrow \ker \phi \xrightarrow{j} N_1 \xrightarrow{\phi} \operatorname{im} \phi \rightarrow 0,$$

where j denote the inclusion morphism. By applying the functor $\text{Hom}_A(., A)$ — which coincides with $\text{Hom}_A^{\text{cont}}(., A)$ on CTC_A when A satisfies **Ded₃**, as shown in Proposition 5.2.4 — to the exact sequence (5.30), we obtain an exact sequence of A -modules:

$$0 \longrightarrow (\text{im } \phi)^\vee \xrightarrow{\phi^\vee} N_1^\vee \xrightarrow{j^\vee} (\ker \phi)^\vee.$$

Since any A -submodule of some object in CP_A is again an object in CP_A , we finally obtain the following exact sequence in CP_A :

$$0 \longrightarrow (\text{im } \phi)^\vee \xrightarrow{\phi^\vee} N_1^\vee \xrightarrow{j^\vee} \text{im } j^\vee \longrightarrow 0.$$

The equivalence of Conditions **E₁** and **E₂** in Proposition 5.4.6 finally shows that the morphism $\phi : N_1 \longrightarrow \text{im } \phi$, already known to be surjective, is also strict. This shows that $\phi : N_1 \longrightarrow N_2$ is strict.

2) For any map $\phi : N_1 \longrightarrow N_2$, its graph

$$\text{Gr } \phi := \{(n, \phi(n)), n \in \mathbb{N}\}$$

is also the inverse image of the diagonal Δ_{N_2} in $N_2 \times N_2$ by the map $(\phi, \text{Id}_{N_2}) : N_1 \times N_2 \longrightarrow N_2 \times N_2$.

When ϕ is continuous, (ϕ, Id_{N_2}) also is continuous, and

$$\text{Gr } \phi = (\phi, \text{Id}_{N_2})^{-1}(\Delta_{N_2})$$

is closed, since the topology of N_2 is Hausdorff and accordingly Δ_{N_2} is closed in $N_2 \longrightarrow N_2 \times N_2$.

Conversely, if ϕ is a continuous A -linear map and if its graph $\text{Gr } \phi$ is closed in $N_1 \times N_2$, then $\text{Gr } \phi$ defines an object of CTC_A by Proposition 5.2.2, and the two projections

$$\text{pr}_i : \text{Gr } \phi \longrightarrow N_i, \quad i = 1, 2$$

are morphisms in CTC_A .

By construction, the map $\text{pr}_1 : \text{Gr } \phi \longrightarrow N_1$ is bijective. According to part 1) of the proof, it is therefore an isomorphism in CTC_A , and therefore $\phi = \text{pr}_2 \circ \text{pr}_1^{-1}$ is finally a morphism in CTC_A . \square

When the ring A is countable — notably when A is the ring of integers of some number field — Proposition 5.4.10 admits an alternative proof that does not rely on the “automatic continuity” of morphisms of A -modules established for general Dedekind rings satisfying **Ded₃**.

Indeed, for any object M of CTC_A , the topology of M may be defined by a complete metric (this follows from the countability assumption in Condition **CTC₂**). If moreover A is countable, the topological space M is separable²² and therefore the additive group $(M, +)$ is a Polish topological group.

Remarkably, the Open Mapping and the Closed Graph Theorems are valid for continuous morphisms of Polish topological groups, by a classical theorem of Banach,²³ and immediately yield the conclusions of Proposition 5.4.10 if A is countable.

Corollary 5.4.11. *When the Dedekind ring A satisfies **Ded₃**, any morphism $\phi : N \longrightarrow N'$ in CTC_A whose cokernel $\text{coker } \phi := N'/\phi(N)$ is a finitely generated A -module is an open map, and therefore a strict morphism.*

Proof. Let (n'_1, \dots, n'_k) be a finite family of elements of N_2 the classes of which generate the A -module $\text{coker } \phi$, and let

$$\tilde{\phi} : N \oplus A^{\oplus k} \longrightarrow N'$$

²²This follows from Propositions 5.2.3 or 5.3.3, which also show that, conversely, if $M \neq 0$ and if the topological space M is separable, A is necessarily countable.

²³See [Ban31]. This theorem now appear as special cases of the “théorème du graphe souslinien”, concerning morphisms of Polish topological groups, presented for instance in [Bou74], Chapitre IX, §6, no. 8, Théorème 4.

the continuous morphism of A -modules defined by the formula:

$$\tilde{\phi}(n, a_1, \dots, a_k) := \phi(n) + a_1 n'_1 + \dots + a_k n'_k$$

for any $n \in N$ and any $(a_1, \dots, a_k) \in A^k$.

By construction, the morphism $\tilde{\phi}$ is surjective, and therefore, according to Proposition 5.4.10, it is strict. In particular, it is an open map. Consequently, the map $\phi : N \rightarrow N'$ — which is the composition of $\tilde{\phi}$ and of the inclusion map $N \hookrightarrow N \oplus A^{\oplus k}$, that is clearly open — is also open. \square

Let us finally indicate that, for any Dedekind ring A that satisfies **Ded**₃, there exist continuous endomorphisms of the A -module $A^{\mathbb{N}}$ which are injective, with dense image, but are not strict (see for instance Proposition 5.5.4, *infra*).

5.4.5. Strict injective morphisms and extensions of scalars. The following proposition is stated for further reference. Its proof is straightforward and left to the reader.

Proposition 5.4.12. *Let $f : N' \rightarrow N$ be a strict injective morphism in CTC_A , and let $(U_i)_{i \in \mathbb{N}}$ be a defining sequence in $\mathcal{U}(N)$.*

1) The sequence $(U'_i)_{i \in \mathbb{N}} := (f^{-1}(U_i))_{i \in \mathbb{N}}$ is a defining sequence in $\mathcal{U}(N')$. Moreover, for every $i \in \mathbb{N}$, the morphism of (finitely generated projective) A -modules

$$f_i : N'_i := N'/U'_i \rightarrow N_i := N/U_i,$$

defined by $f_i(x' + U'_i) = f(x') + U_i$ for any $x' \in N'$, is injective.

When the topological A -modules N and N' are identified with the projective limits $\varprojlim_i N_i$ and $\varprojlim_i N'_i$, the morphism $f : N' \rightarrow N$ gets identified with the morphism $\varprojlim_i f_i$.

2) For any field extension L of the fraction field K of A , the topological L -modules M'_L and M_L may be identified with $\varprojlim_i N_{i,L}$ and $\varprojlim_i N'_{i,L}$ and the morphism $f_L : N'_L \rightarrow N_L$ with the morphism $\varprojlim_i f_{i,L}$.

In particular, like the morphisms $f_{i,L}$, the morphism f_L is injective. \square

Let us indicate that, with the notation of Proposition 5.4.12, if the injective morphism $f : N' \rightarrow N$ is not assumed to be strict, its base change f_K may be non-injective. (For instance, the morphism $\beta_1 : \mathbb{Z}^{\mathbb{N}} \rightarrow \mathbb{Z}^{\mathbb{N}}$ in $CTC_{\mathbb{Z}}$ considered in Proposition 5.5.1, 1), *infra* is injective, but $\beta_{1\mathbb{Q}} : \mathbb{Q}^{\mathbb{N}} \rightarrow \mathbb{Q}^{\mathbb{N}}$ admits the line $\mathbb{Q} \cdot (p^{-k})_{k \in \mathbb{N}}$ as kernel.)

5.5. Examples. In this section, we discuss some simple examples of objects and morphisms in the categories CP_A and CTC_A when A is the ring \mathbb{Z}_p or \mathbb{Z} .

These examples should make clear that non-strict morphisms in the categories $CTC_{\mathbb{Z}_p}$ and $CTC_{\mathbb{Z}}$ are not “pathologies” but occur “naturally”, and that exact sequences in the categories $CP_{\mathbb{Z}}$ and $CTC_{\mathbb{Z}}$ and their duality properties must be handled with some care.

We denote by p a prime number and by $|\cdot|_p$ the p -adic norm of \mathbb{Z}_p on the ring \mathbb{Z}_p of p -adic integers, defined by $|p^n u|_p := p^{-n}$ for every $n \in \mathbb{N}$ and every $u \in \mathbb{Z}_p^{\times}$.

5.5.1. Subobjects and duality in $CP_{\mathbb{Z}}$ and $CTC_{\mathbb{Z}}$. Besides Condition **E**₂, we may consider the following condition:

E₂ : *The morphism α is injective and its image is saturated in M' .*

Similarly, besides Condition **F**₁, we may consider the following condition:

F₁ : *The morphism β is injective and its image is closed and saturated in N_2 .*

Clearly, Condition **E**₂ implies Condition **E**₂, and Condition **F**₁ implies Condition **F**₁.

However, the converse implications do not hold in general. This is demonstrated, when $A = \mathbb{Z}$, by Proposition 5.5.1 below.

Let $(\epsilon_n)_{n \in \mathbb{N}}$ denote the canonical basis of $\mathbb{Z}^{(\mathbb{N})}$, defined by $\epsilon_n(k) := \delta_{nk}$ for every $(n, k) \in \mathbb{N}^2$. We shall identify the dual in $CTC_{\mathbb{Z}}$ of the object $M := \mathbb{Z}^{(\mathbb{N})}$ of $CP_{\mathbb{Z}}$ with the \mathbb{Z} -module $N := \mathbb{Z}^{\mathbb{N}}$ equipped with the product topology of the discrete topology on \mathbb{Z} , by means of the map:

$$(5.31) \quad \begin{aligned} \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^{(\mathbb{N})}, \mathbb{Z}) &\xrightarrow{\sim} \mathbb{Z}^{\mathbb{N}} \\ \xi &\longmapsto (\xi(\epsilon_n))_{n \in \mathbb{N}}. \end{aligned}$$

Proposition 5.5.1. 1) If we define two morphisms of \mathbb{Z} -modules $\alpha_1 : \mathbb{Z}^{(\mathbb{N})} \rightarrow \mathbb{Z}^{(\mathbb{N})}$ and $\pi_1 : \mathbb{Z}^{(\mathbb{N})} \rightarrow \mathbb{Z}[1/p]$ by

$$\alpha_1(\epsilon_n) := \epsilon_n - p\epsilon_{n+1} \text{ and } \pi_1(\epsilon_n) := \frac{1}{p^n}$$

for every $n \in \mathbb{N}$, then α_1 and π_1 fit into the following short exact sequence:

$$(5.32) \quad 0 \longrightarrow \mathbb{Z}^{(\mathbb{N})} \xrightarrow{\alpha_1} \mathbb{Z}^{(\mathbb{N})} \xrightarrow{\pi_1} \mathbb{Z}[1/p] \longrightarrow 0.$$

Moreover the dual morphism $\beta_1 := \alpha_1^\vee \in \text{Hom}_{\mathbb{Z}}^{\text{cont}}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z}^{\mathbb{N}})$ is injective, and if we define a \mathbb{Z} -linear map $\sigma_1 : \mathbb{Z}^{\mathbb{N}} \rightarrow \mathbb{Z}_p/\mathbb{Z}$ by

$$\sigma_1((y_k)_{k \in \mathbb{N}}) := \left[\sum_{k \in \mathbb{N}} p^k y_k \right],$$

then the following diagram is a short exact sequence:

$$(5.33) \quad 0 \longrightarrow \mathbb{Z}^{\mathbb{N}} \xrightarrow{\beta_1} \mathbb{Z}^{\mathbb{N}} \xrightarrow{\sigma_1} \mathbb{Z}_p/\mathbb{Z} \longrightarrow 0.$$

2) If we define two morphisms of \mathbb{Z} -modules $\alpha_2 : \mathbb{Z}^{(\mathbb{N})} \rightarrow \mathbb{Z}^{(\mathbb{N})}$ and $\pi_2 : \mathbb{Z}^{(\mathbb{N})} \rightarrow \mathbb{Z}[1/p]/\mathbb{Z}$ by

$$\begin{aligned} \alpha_2(\epsilon_n) &:= -p\epsilon_0 && \text{when } n = 0, \\ &:= \epsilon_{n-1} - p\epsilon_n && \text{when } n \geq 1, \end{aligned}$$

and by

$$\pi_2(\epsilon_n) := \left[\frac{1}{p^{n+1}} \right] \text{ for every } n \in \mathbb{N},$$

then α_2 and π_2 fit into the following short exact sequence:

$$(5.34) \quad 0 \longrightarrow \mathbb{Z}^{(\mathbb{N})} \xrightarrow{\alpha_2} \mathbb{Z}^{(\mathbb{N})} \xrightarrow{\pi_2} \mathbb{Z}[1/p]/\mathbb{Z} \longrightarrow 0.$$

Moreover the dual morphism $\beta_2 := \alpha_2^\vee$ in $\text{Hom}_{\mathbb{Z}}^{\text{cont}}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z}^{\mathbb{N}})$ is injective, and if we define a continuous \mathbb{Z} -linear map $\sigma_2 : \mathbb{Z}^{\mathbb{N}} \rightarrow \mathbb{Z}_p$ by

$$\sigma_2((y_k)_{k \in \mathbb{N}}) := \sum_{k \in \mathbb{N}} p^k y_k,$$

then the following diagram is a short exact sequence:

$$(5.35) \quad 0 \longrightarrow \mathbb{Z}^{\mathbb{N}} \xrightarrow{\beta_2} \mathbb{Z}^{\mathbb{N}} \xrightarrow{\sigma_2} \mathbb{Z}_p \longrightarrow 0.$$

Observe that α_1 satisfies \mathbf{E}'_2 and not \mathbf{E}_2 , since β_1 does not satisfy \mathbf{E}_1 , and that β_2 satisfies \mathbf{F}'_1 and not \mathbf{F}_1 , since α_2 does not satisfies \mathbf{F}_2 .

As already mentioned, Proposition 5.5.1 also demonstrates how the categories $CP_{\mathbb{Z}}$ and $CTC_{\mathbb{Z}}$ are “badly behaved” with respect to quotients.

Proof. 1) We use the identification of \mathbb{Z} -modules

$$\begin{aligned} \mathbb{Z}^{(\mathbb{N})} &\xrightarrow{\sim} \mathbb{Z}[X] && \text{and} && \mathbb{Z}^{\mathbb{N}} &\xrightarrow{\sim} \mathbb{Z}[[X]] \\ \epsilon_n &\longmapsto X^n && && (y_k)_{k \in \mathbb{N}} &\longmapsto \sum_{k \in \mathbb{N}} y_k X^k. \end{aligned}$$

The isomorphism (5.31) becomes the isomorphism

$$\text{Hom}_{\mathbb{Z}}(\mathbb{Z}[X], \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}[[X]]$$

induced by the residue pairing

$$(5.36) \quad \begin{array}{ccc} \mathbb{Z}[[X]] \times \mathbb{Z}[X] & \longrightarrow & \mathbb{Z} \\ (f, P) & \longmapsto & \text{Res}_{X=0}[f(X)P(1/X)dX/X]. \end{array}$$

The diagram (5.32) may be written

$$(5.37) \quad 0 \longrightarrow \mathbb{Z}[X] \xrightarrow{\alpha_1} \mathbb{Z}[X] \xrightarrow{\pi_1} \mathbb{Z}[1/p] \longrightarrow 0$$

where

$$(5.38) \quad \alpha_1(P) := (1 - pX)P$$

and

$$\pi_1(P) = P(1/p).$$

The exactness of (5.37), hence of (5.32), follows from the basic properties of polynomials with integer coefficients.

Moreover the expressions (5.38) for α_1 and (5.36) for the duality pairing between $\mathbb{Z}[[X]]$ and $\mathbb{Z}[X]$ show that the dual morphism

$$\beta_1 := \alpha_1^\vee : \mathbb{Z}[[X]] \longrightarrow \mathbb{Z}[[X]]$$

is given by

$$(5.39) \quad \beta_1(f) = (1 - p/X)f + pf(0)/X.$$

The vanishing of $\beta_1(f)$ therefore implies the equality in $\mathbb{Q}[[X]]$:

$$f(X) = f(0)(1 - p^{-1}X)^{-1} = f(0) \sum_{k \in \mathbb{N}} p^{-k} X^k.$$

Clearly, the only f in $\mathbb{Z}[[X]]$ satisfying this condition is $f = 0$, and accordingly β_1 is injective.

To complete the proof of the exactness of (5.33), we shall use the following easy lemma:

Lemma 5.5.2. *The evaluation morphism*

$$(5.40) \quad \begin{array}{ccc} \eta : \mathbb{Z}[[X]] & \longrightarrow & \mathbb{Z}_p \\ f & \longmapsto & f(p) \end{array}$$

is surjective. Its kernel is $(X - p)\mathbb{Z}[[X]]$.

Proof of Lemma 5.5.2. The surjectivity of η is clear, and the inclusion $(X - p)\mathbb{Z}[[X]] \subset \ker \eta$ also.

To establish the converse inclusion, observe that the kernel of the evaluation map

$$\eta_{\mathbb{Z}_p} : \begin{array}{ccc} \mathbb{Z}_p[[X]] & \longrightarrow & \mathbb{Z}_p \\ f & \longmapsto & f(p) \end{array}$$

is $(X - p)\mathbb{Z}_p[[X]]$, say by the Weierstrass preparation theorem.

Therefore any element f of $\mathbb{Z}[[X]]$ in the kernel of η may be written

$$(5.41) \quad f = (X - p)g$$

for some g in $\mathbb{Z}_p[[X]]$. The polynomial $(X - p) = -p(1 - X/p)$ is a unit in $\mathbb{Z}[1/p][[X]]$, and the equation (5.41) shows that the series g , seen as an element of $\mathbb{Q}_p[[X]]$, actually belongs to $\mathbb{Z}[1/p][[X]]$. Consequently g belongs to $\mathbb{Z}_p[[X]] \cap \mathbb{Z}[1/p][[X]] = \mathbb{Z}[[X]]$. \square

The morphism $\sigma_1 : \mathbb{Z}[[X]] \longrightarrow \mathbb{Z}_p/\mathbb{Z}$ maps $g \in \mathbb{Z}[[X]]$ to the class of $g(p)$ in \mathbb{Z}_p/\mathbb{Z} . It is clearly surjective, and to establish the exactness of (5.33), we are indeed left to show that, for any $g \in \mathbb{Z}[[X]]$, the following two conditions are equivalent:

(i) there exists $f \in \mathbb{Z}[[X]]$ such that

$$g = (1 - p/X)f + pf(0)/X;$$

(ii) the element $g(p)$ of \mathbb{Z}_p belongs to \mathbb{Z} .

When (i) holds, then $g(p) = f(0)$ and therefore (ii) also holds. Conversely, when (ii) holds, then $g - g(p)$ is an element of $\mathbb{Z}[[X]]$ in the kernel of the evaluation morphism (5.40) and, for some $h \in \mathbb{Z}[[X]]$, we have

$$g - g(p) = (X - p)h.$$

Therefore condition (i) is satisfied by $f := Xh + g(p)$.

2) Similarly the exact sequence (5.34) and (5.35) may be written as

$$(5.42) \quad 0 \longrightarrow \mathbb{Z}[X] \xrightarrow{\alpha_2} \mathbb{Z}[X] \xrightarrow{\pi_2} \mathbb{Z}[1/p]/\mathbb{Z} \longrightarrow 0,$$

where

$$\alpha_2(P) := \frac{1 - pX}{X}P - \frac{P(0)}{X} = \frac{P - P(0)}{X} - pP$$

and

$$\pi_2(P) = (1/p) \cdot P(1/p) \pmod{\mathbb{Z}},$$

and as

$$(5.43) \quad 0 \longrightarrow \mathbb{Z}[[X]] \xrightarrow{\beta_2} \mathbb{Z}[[X]] \xrightarrow{\sigma_2} \mathbb{Z}_p \longrightarrow 0,$$

where

$$\beta_2(f) = \alpha_2(f) := (X - p)f \text{ and } \sigma_2(f) := f(p).$$

We leave the proof of the exactness of (5.42) as an elementary exercise on polynomials with integer coefficients. The exactness of (5.43) is basically the content of Lemma 5.5.2. \square

5.5.2. A non-strict bijective morphism in $CTC_{\mathbb{Z}_p}$. We want to point out that, when the base ring A satisfies **Ded₂** — that is, when A a complete discrete valuation ring — the Open Mapping and the Closed Graph Theorems (as stated in Proposition 5.4.10 for a Dedekind ring A satisfying **Ded₃**) do not hold.

For definiteness, let us assume that $A = \mathbb{Z}_p$, and let N be the object of $CTC_{\mathbb{Z}_p}$ defined as $N := \mathbb{Z}_p^{\mathbb{N}}$ equipped with the product of the discrete topology on each factor \mathbb{Z}_p .

Lemma 5.5.3. *Let $(\xi_n)_{n \in \mathbb{N}}$ be an element of $\mathbb{Z}_p^{\mathbb{N}}$ such that $\lim_{n \rightarrow +\infty} |\xi_n|_p = 0$. For any $x := (x_n) \in N = \mathbb{Z}_p^{\mathbb{N}}$, the series $\xi(x) := \sum_{n \in \mathbb{N}} \xi_n x_n$ converges in \mathbb{Z}_p equipped with $|\cdot|_p$, and defines a \mathbb{Z}_p -linear map $\xi : N \longrightarrow \mathbb{Z}_p$.*

The map ξ belongs to $\text{Hom}_{CTC_{\mathbb{Z}_p}}(N, \mathbb{Z}_p)$ if and only if $(\xi_n)_{n \in \mathbb{N}}$ belongs to $\mathbb{Z}_p^{(\mathbb{N})}$.

The graph $\text{Gr } \xi$ of ξ is closed in the topological module $N \oplus \mathbb{Z}_p$ of $CTC_{\mathbb{Z}_p}$.

Proof. All the assertions are immediate, but possibly the last one.

To prove that $\text{Gr } \xi$ of ξ is closed in $N \oplus \mathbb{Z}_p = \mathbb{Z}_p^{\mathbb{N}} \oplus \mathbb{Z}_p$ equipped with the product of the discrete topology on each factor \mathbb{Z}_p , consider x in N and a in \mathbb{Z}_p such that $(x, a) \notin \text{Gr } \xi$, or equivalently, such that $\xi(x) \neq a$.

There exists a positive integer n_0 such that, for every $n \in \mathbb{N}_{>n_0}$,

$$|\xi_n|_p < |\xi(x) - a|_p.$$

Then $U := \{0\}^{\{0, \dots, n_0\}} \times \mathbb{Z}_p^{\mathbb{N}_{>n_0}}$ is an open neighborhood of 0 in $N = \mathbb{Z}_p^{\mathbb{N}}$ such that, for any $u \in U$,

$$|\xi(u)|_p < |\xi(x) - a|_p.$$

Consequently, for any $\tilde{x} \in x + U$,

$$|\xi(\tilde{x}) - a|_p = |\xi(x) - a|_p \neq 0,$$

and $(x, a) + U \oplus \{0\}$ is an open neighborhood of (x, a) in $N \oplus \mathbb{Z}_p$ disjoint of $\text{Gr } \xi$. \square

Let us keep the notation of Lemma 5.5.3.

According to Proposition 5.2.2, the A -module $\mathrm{Gr} \xi$, equipped with the topology induced by the one of $N \oplus \mathbb{Z}_p$, becomes an object of $CTC_{\mathbb{Z}_p}$. Clearly the first projection

$$\mathrm{pr}_{1|\mathrm{Gr} \xi} : \mathrm{Gr} \xi \longrightarrow N$$

defines a continuous bijective morphism of topological A -modules. By construction, it is a homeomorphism — or equivalently, a strict morphism — precisely when $\xi : N \longrightarrow \mathbb{Z}_p$ is continuous.

This shows that, for any $(\xi_n)_{n \in \mathbb{N}}$ in $\mathbb{Z}_p^{\mathbb{N}} \setminus \mathbb{Z}_p^{(\mathbb{N})}$ such that $\lim_{n \rightarrow +\infty} |\xi_n|_p = 0$, the map $\mathrm{pr}_{1|\mathrm{Gr} \xi}$ is a non-strict bijective morphism in $\mathrm{Hom}_{CTC_{\mathbb{Z}_p}}(\mathrm{Gr} \xi, N)$.

5.5.3. Non-strict injective morphisms in CTC_A when A satisfies **Ded₃.** The construction in Proposition 5.5.1, 2), may easily be extended to a more general setting and provides, for any Dedekind ring A that satisfies **Ded₃**, examples of non-strict injective morphisms in CTC_A that are not strict.

The ring $A[[X]]$ of formal series with coefficients in A will be equipped with its natural prodiscrete topology (say, defined by its identification with $\varprojlim_n A[X]/(X^n)$). Then it becomes an object of CTC_A .

By mimicking the arguments in the proof of Proposition 5.5.1, 2), one easily establishes the following proposition. (To prove its assertion 3), choose some non-zero prime ideal \mathfrak{p} dividing a , and consider the evaluation map $g \mapsto g(a)$ from A to the \mathfrak{p} -adic completion $\hat{A}_{\mathfrak{p}}$ of A . On the image $\phi_a(A[[X]])$ of ϕ_a , this evaluation map takes values in A .) We leave the details of its proof to the reader.

Proposition 5.5.4. *For any $a \in A$, we define a continuous A -linear map*

$$\phi_a : A[[X]] \longrightarrow A[[X]]$$

by letting

$$\phi_a(f) := (1 - a/X)f + af(0)/X = f - a(f - f(0))/X.$$

- 1) *The map ϕ_a is injective if and only if $a \notin A^\times$.*
- 2) *For any a in A , the map ϕ_a sends $A[X]$ bijectively onto $A[X]$. In particular, its image $\phi_a(A[[X]])$ is dense in $A[[X]]$.*
- 3) *If A satisfies **Ded₃**, then, for any $a \in A \setminus (A^\times \cup \{0\})$, the map ϕ_a is not surjective.* □

6. IND- AND PRO-HERMITIAN VECTOR BUNDLES OVER ARITHMETIC CURVES

In this section, we introduce diverse categories of infinite dimensional hermitian vector bundles over an arithmetic curve $\mathrm{Spec} \mathcal{O}_K$ defined by the ring of integers \mathcal{O}_K of some number field K . These categories are constructed from the categories CP_A and CTC_A investigated in the previous sections, specialized to the case of the Dedekind ring $A = \mathcal{O}_K$ (of type **Ded₃**), by enriching the objects and the morphisms by “hermitian data”.

In the applications to Diophantine geometry, we will be mainly interested in *pro-hermitian* vector bundles: their underlying “algebraic” objects will be the topological \mathcal{O}_K -modules in $CTC_{\mathcal{O}_K}$. In the basic case where $\mathcal{O}_K = \mathbb{Z}$, these are precisely the pro-euclidean lattices described in the Introduction (see 1.1.1). As already indicated in *loc. cit.*, they admit alternative descriptions, either (i) in terms of objects \hat{E} of $CTC_{\mathcal{O}_K}$ and of Hilbert spaces E_σ^{Hilb} densely embedded in the completed tensor products $\hat{E}_\sigma \simeq \hat{E} \hat{\otimes}_\sigma \mathbb{C}$ associated to the diverse complex embeddings $\sigma : K \hookrightarrow \mathbb{C}$, or (ii) in terms of projective systems

$$\overline{E}_\bullet : \overline{E}_0 \xleftarrow{q_0} \overline{E}_1 \xleftarrow{q_1} \dots \xleftarrow{q_{i-1}} \overline{E}_i \xleftarrow{q_i} \overline{E}_{i+1} \xleftarrow{q_{i+1}} \dots$$

of surjective admissible morphisms of hermitian vector bundles over $\mathrm{Spec} \mathcal{O}_K$.

The equivalence of these descriptions is elementary, but quite useful in practice. Other constructions described in this section are mostly formal, and their details could be skipped at first reading.

The last paragraphs of this section are devoted to diverse examples, that are quite elementary but should convey some feeling of the “concrete significance” of pro-hermitian vector bundles and of the technical subtleties one may encounter when handling them.

Notably, in Subsection 6.6, we introduce the “arithmetic Hardy spaces” $\widehat{H}(R)$ and the “arithmetic Bergman spaces” $\widehat{B}(R)$. These pro-euclidean lattices constitute the archetypes of the pro-hermitian vector bundles over arithmetic curves that we shall investigate in the sequel, when applying the formalism developed here to study the interaction of complex analytic geometry and formal geometry (over the integers) in diverse Diophantine settings.

Besides, in Subsection 6.7 we show, by explicit examples, that the injectivity or surjectivity properties of the morphisms of topological \mathcal{O}_K -modules and of complex Fréchet and Hilbert spaces underlying some morphism of pro-hermitian vector bundles are in general rather subtly related.

We denote by K a number field, by \mathcal{O}_K its ring of integers, and by $\pi : \text{Spec } \mathcal{O}_K \longrightarrow \text{Spec } \mathbb{Z}$ the morphism of schemes from $\text{Spec } \mathcal{O}_K$ to $\text{Spec } \mathbb{Z}$.

6.1. Definitions.

6.1.1. Ind-hermitian vector bundles. We define an *ind-hermitian vector bundle* over $\text{Spec } \mathcal{O}_K$ as a pair

$$\overline{F} := (F, (\|\cdot\|_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}})$$

where F is an object of $CP_{\mathcal{O}_K}$ — namely a countably generated projective \mathcal{O}_K -module — and $(\|\cdot\|_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}}$ is a family of prehilbertian norms on the complex vector spaces

$$F_\sigma := F \otimes_\sigma \mathbb{C}$$

deduced from the \mathcal{O}_K -module E by the base change $\sigma : \mathcal{O}_K \longrightarrow \mathbb{C}$. Moreover, the family $(\|\cdot\|_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}}$ is required to be invariant under complex conjugation.²⁴

An *isometric isomorphism* $\phi : \overline{F} \longrightarrow \overline{F}'$ between two ind-hermitian vector bundles $\overline{F} := (F, (\|\cdot\|_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}})$ and $\overline{F}' := (F', (\|\cdot\|'_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}})$ over $\text{Spec } \mathcal{O}_K$ is an isomorphism of \mathcal{O}_K -modules $\phi : F \xrightarrow{\sim} F'$ such that, for every embedding $\sigma : K \hookrightarrow \mathbb{C}$, the \mathbb{C} -linear isomorphism $\phi_\sigma : F_\sigma \xrightarrow{\sim} F'_\sigma$ is isometric with respect to the norms $\|\cdot\|_\sigma$ and $\|\cdot\|'_\sigma$.

6.1.2. Pro-hermitian vector bundles. By definition, a *pro-hermitian vector bundle* over $\text{Spec } \mathcal{O}_K$ is the data

$$\widehat{E} := (\widehat{E}, (\overline{E}_U)_{U \in \mathcal{U}(\widehat{E})})$$

of an object \widehat{E} of $CTC_{\mathcal{O}_K}$ and, for any open saturated \mathcal{O}_K -submodule U of \widehat{E} , of a structure of hermitian vector bundle over $\text{Spec } \mathcal{O}_K$

$$\overline{E}_U := (E_U, (\|\cdot\|_{U,\sigma})_{\sigma:K \hookrightarrow \mathbb{C}})$$

on the finitely generated projective \mathcal{O}_K -module $E_U := \widehat{E}/U$.

²⁴Namely, for any field embedding $\sigma : K \hookrightarrow \mathbb{C}$, the norm $\|\cdot\|_\sigma$ on F_σ and the norm $\|\cdot\|_{\overline{\sigma}}$ on $F_{\overline{\sigma}}$ attached to the complex conjugate embedding $\overline{\sigma}$ coincide through the \mathbb{C} -antilinear isomorphism

$$\begin{aligned} F \otimes_\sigma \mathbb{C} &\simeq F \otimes_{\overline{\sigma}} \mathbb{C} \\ e \otimes \lambda &\longmapsto \overline{e} \otimes \overline{\lambda} := e \otimes \overline{\lambda}. \end{aligned}$$

In other words, $\|\overline{v}\|_{\overline{\sigma}} = \|v\|_\sigma$ for every $v \in F_\sigma$.

Moreover, for any two open saturated \mathcal{O}_K -submodules U and U' of \widehat{E} such that $U \subset U'$, the surjective morphism of \mathcal{O}_K -modules $p_{U'U} : E_U \rightarrow E_{U'}$ is required to define a surjective *admissible* morphism of hermitian vector bundles from \overline{E}_U onto $\overline{E}_{U'}$.

An *isometric isomorphism* $\psi : \widehat{E} \rightarrow \widehat{E}'$ between two pro-hermitian vector bundles

$$\widehat{E} = (\widehat{E}, (\overline{E}_U)_{U \in \mathcal{U}(\widehat{E})})$$

and

$$\widehat{E}' := (\widehat{E}', (\overline{E}'_{U'})_{U' \in \mathcal{U}(\widehat{E}')})$$

is an isomorphism $\psi : \widehat{E} \xrightarrow{\sim} \widehat{E}'$ of topological²⁵ \mathcal{O}_K -modules such that, for any U in $\mathcal{U}(\widehat{E})$, of image $U' := \psi(U)$ (necessarily in $\mathcal{U}(\widehat{E}')$), the induced isomorphism of \mathcal{O}_K -modules

$$\psi_U : E_U := \widehat{E}/U \rightarrow E'_{U'} := \widehat{E}'/U'$$

defines an isometric isomorphism of hermitian vector bundles from \overline{E}_U onto $\overline{E}'_{U'}$.

6.1.3. The complex topological vector spaces associated to a pro-hermitian vector bundle. Let $\widehat{E} := (\widehat{E}, (\overline{E}_U)_{U \in \mathcal{U}(\widehat{E})})$ be a pro-hermitian vector bundle over $\text{Spec } \mathcal{O}_K$, as above, and let $\sigma : K \hookrightarrow \mathbb{C}$ be a field embedding.

(i) We may apply the functor

$$\cdot \hat{\otimes}_{\mathcal{O}_K, \sigma} : CTC_{\mathcal{O}_K} \rightarrow CTC_{\mathbb{C}}$$

to the topological \mathcal{O}_K -module \widehat{E} . We thus define the completed tensor product

$$\widehat{E}_\sigma := \widehat{E} \hat{\otimes}_{\mathcal{O}_K, \sigma} \mathbb{C}.$$

By its very definition, \widehat{E}_σ may be identified with an inverse limit of finite dimensional complex vector spaces equipped with the discrete topology:

$$\widehat{E}_\sigma \simeq \varprojlim_{U \in \mathcal{U}(\widehat{E})} E_{U, \sigma}.$$

Besides this “pro-discrete” topology, which makes it an object of $CTC_{\mathbb{C}}$, the complex vector space \widehat{E}_σ also admits a canonical separated and locally convex topology : it is defined by taking the projective limit $\varprojlim_{U \in \mathcal{U}(\widehat{E})} E_{U, \sigma}$ that defines \widehat{E}_σ in the category of locally convex complex vector spaces, when each finite dimensional complex vector space $E_{U, \sigma}$ is equipped with its usual (separated and locally convex) topology.

Equipped with this topology, \widehat{E}_σ is a nuclear Fréchet space. Actually, in the category $CTC_{\mathbb{C}}$, there exists an isomorphism $\phi : \widehat{E}_\sigma \xrightarrow{\sim} \mathbb{C}^I$ for some countable set I (where \mathbb{C}^I is equipped with the product of the discrete topology on each factor \mathbb{C}), and any such isomorphism is an isomorphism of complex locally convex vector spaces, when \widehat{E}_σ (resp., \mathbb{C}^I) is equipped with its natural locally convex topology (resp., with the product of the usual topology on each factor \mathbb{C}).

From now on, the completed tensor products \widehat{E}_σ associated to some pro-hermitian vector bundle \widehat{E} (or more generally to some object \widehat{E} in $CTC_{\mathcal{O}_K}$) will be always be endowed with its canonical topology of complex Fréchet space.

We shall denote by

$$p_U : \widehat{E} \rightarrow \widehat{E}/U =: E_U$$

the quotient map, and by

$$p_{U, \sigma} : \widehat{E}_\sigma \rightarrow E_{U, \sigma}$$

²⁵According to Proposition 5.2.4, any isomorphism $\psi : \widehat{E} \xrightarrow{\sim} \widehat{E}'$ of \mathcal{O}_K -modules is actually an isomorphism of topological \mathcal{O}_K -modules.

its “completed complexification”.

(ii) Observe that the hermitian vector spaces $\overline{E}_{U,\sigma}$ and the “admissible” \mathbb{C} -linear maps $p_{U'U,\sigma} : E_{U,\sigma} \rightarrow E_{U',\sigma}$ also constitute a projective system in the category the objects of which are the complex normed spaces and the morphisms, the continuous linear maps of operator norm ≤ 1 .

This projective system admits a limit in this category, that may be described as follows.

Its underlying \mathbb{C} -vector space is the subspace of

$$\begin{aligned} \widehat{E}_\sigma &:= \varprojlim_{U \in \mathcal{U}(\widehat{E})} E_{U,\sigma} \\ &= \left\{ (x_U)_{U \in \mathcal{U}(\widehat{E})} \in \prod_{U \in \mathcal{U}(\widehat{E})} E_{U,\sigma} \mid \text{for any } (U, U') \in \mathcal{U}(\widehat{E})^2, U \subset U' \implies p_{U'U}(x_U) = x_{U'} \right\} \end{aligned}$$

defined by “uniformly bounded” elements, namely:

$$E_\sigma^{\text{Hilb}} := \varprojlim_U^{\text{Hilb}} E_{U,\sigma} = \left\{ (x_U)_{U \in \mathcal{U}(\widehat{E})} \in \varprojlim_U E_{U,\sigma} \mid \sup_{U \in \mathcal{U}(\widehat{E})} \|x_U\|_{\overline{E}_{U,\sigma}} < +\infty \right\}.$$

Its norm is the norm $\|\cdot\|_{E_\sigma^{\text{Hilb}}}$ defined by the equality

$$(6.1) \quad \|x\|_{E_\sigma^{\text{Hilb}}} := \sup_{U \in \mathcal{U}(\widehat{E})} \|x_U\|_{\overline{E}_{U,\sigma}}$$

for any element $x = (x_U)_{U \in \mathcal{U}(\widehat{E})}$ in E_σ^{Hilb} . Actually $\|x_U\|_{\overline{E}_{U,\sigma}}$ is a non-decreasing function of $U \in \mathcal{U}(\widehat{E})$, and therefore we also have:

$$\|x\|_{E_\sigma^{\text{Hilb}}} = \lim_{U \in \mathcal{U}(\widehat{E})} \|x_U\|_{\overline{E}_{U,\sigma}}.$$

The following proposition is a straightforward consequence of the definitions, and its proof is left to the reader:

Proposition 6.1.1. *Equipped with the norm $\|\cdot\|_{E_\sigma^{\text{Hilb}}}$, the complex vector space $\widehat{E}_\sigma^{\text{Hilb}}$ becomes a separable Hilbert space. For any $U \in \mathcal{U}(\widehat{E})$, the map*

$$p_{U,\sigma|E_\sigma^{\text{Hilb}}} : E_\sigma^{\text{Hilb}} \rightarrow E_{U,\sigma}$$

is a “co-isometry”. In other words, the hermitian norm $\|\cdot\|_{\overline{E}_{U,\sigma}}$ coincides with the quotient norm deduced from the Hilbertian norm $\|\cdot\|_{E_\sigma^{\text{Hilb}}}$ by means of the surjective \mathbb{C} -linear map $p_{U,\sigma|E_\sigma^{\text{Hilb}}}$.

Moreover, when $\widehat{E}_\sigma^{\text{Hilb}}$ is equipped with its topology of Hilbert space, and \widehat{E}_σ with its canonical topology of separated locally convex complex vector space, the inclusion morphism

$$i_\sigma : E_\sigma^{\text{Hilb}} \rightarrow \widehat{E}_\sigma$$

is continuous with dense image. □

Finally observe that the constructions (i) and (ii) are clearly compatible with isometric isomorphisms of pro-hermitian vector bundles.

6.1.4. An alternative description of pro-hermitian vector bundles. (i) The previous constructions lead us to the following alternative definition of pro-hermitian vector bundles over $\text{Spec } \mathcal{O}_K$, which turns out to be more flexible as their initial definition in terms of projective systems of (finite dimensional) hermitian vector bundles.

We may define a pro-hermitian vector bundle over $\text{Spec } \mathcal{O}_K$ as the data

$$(6.2) \quad \widehat{E} := (\widehat{E}, (E_\sigma^{\text{Hilb}}, \|\cdot\|_\sigma, i_\sigma)_{\sigma: K \hookrightarrow \mathbb{C}}),$$

where \widehat{E} is an object of $CTC_{\mathcal{O}_K}$ and where, for every field embedding $\sigma : K \hookrightarrow \mathbb{C}$, E_σ^{Hilb} denotes a complex Hilbert space, $\|\cdot\|_\sigma$ its norm, and $i_\sigma : E_\sigma^{\text{Hilb}} \longrightarrow \widehat{E}_\sigma$ a continuous injective \mathbb{C} -linear map with dense image.

The data

$$((E_\sigma^{\text{Hilb}}, \|\cdot\|_\sigma, i_\sigma))_{\sigma: K \hookrightarrow \mathbb{C}}$$

are required to be compatible with complex conjugation. Namely, one requires the existence, for every $\sigma : K \hookrightarrow \mathbb{C}$, of a \mathbb{C} -antilinear bijective isometry $\gamma_\sigma : E_\sigma^{\text{Hilb}} \xrightarrow{\sim} E_{\bar{\sigma}}^{\text{Hilb}}$ such that the following relation hold, where $\bar{\cdot}$ denotes the \mathbb{C} -antilinear isomorphism from $\widehat{E}_\sigma := \widehat{E} \hat{\otimes}_\sigma \mathbb{C}$ onto $\widehat{E}_{\bar{\sigma}} := \widehat{E} \hat{\otimes}_{\bar{\sigma}} \mathbb{C}$ deduced from the complex conjugation on \mathbb{C} :

$$(6.3) \quad i_{\bar{\sigma}} \circ \gamma_\sigma = \bar{\cdot} \circ i_\sigma.$$

When they exist, the maps γ_σ are uniquely determined by the relations (6.3). Moreover, they are easily seen to exist when one is given a pro-hermitian vector bundle $\widehat{E} := (\widehat{E}, (\overline{E}_U)_{U \in \mathcal{U}(\widehat{E})})$ in the sense of paragraph 6.1.2, and when E_σ^{Hilb} , $\|\cdot\|_\sigma := \|\cdot\|_{E_\sigma^{\text{Hilb}}}$, and i_σ are defined as in paragraph 6.1.3, (ii).

Conversely, starting from the data (6.2), for every $U \in \mathcal{U}(\widehat{E})$, one defines a structure of hermitian vector bundle

$$\overline{E}_U := (E_U, (\|\cdot\|_{U,\sigma})_{\sigma: K \hookrightarrow \mathbb{C}})$$

on the finitely generated projective \mathcal{O}_K -module $E_U := \widehat{E}/U$ by defining the norm $\|\cdot\|_{U,\sigma}$ as the quotient norm deduced from the Hilbert norm $\|\cdot\|_\sigma$ on E_σ^{Hilb} by requiring the \mathbb{C} -linear maps

$$p_{U,\sigma} \circ i_\sigma : E_\sigma^{\text{Hilb}} \longrightarrow E_{U,\sigma}$$

to be co-isometry. (Observe that the density of $i_\sigma(E_\sigma^{\text{Hilb}})$ in \widehat{E}_σ precisely means that this map is surjective for any $U \in \mathcal{U}(\widehat{E})$.)

In this way, one constructs a pro-hermitian vector bundle $\widehat{E} := (\widehat{E}, (\overline{E}_U)_{U \in \mathcal{U}(\widehat{E})})$ in the sense of paragraph 6.1.2 from the data (6.2).

The reader will easily check that these two constructions are inverse of each other.

(ii) When dealing with pro-hermitian vector bundles defined by data of type (6.2), we shall occasionally write $i_{\widehat{E}}^\sigma$ instead of i_σ to make the dependence on \widehat{E} explicit, notably when discussing “concrete” examples of pro-hermitian vector bundles occurring in Diophantine geometry.

Conversely, when investigating the general properties of pro-hermitian vector bundles shall also sometimes avoid to name explicitly the morphisms i_σ and γ_σ , and identify E_σ^{Hilb} to its image by i_σ . Accordingly, a pro-hermitian vector bundle will be often denoted by

$$\widehat{E} := (\widehat{E}, (E_\sigma^{\text{Hilb}}, \|\cdot\|_\sigma)_{\sigma: K \hookrightarrow \mathbb{C}}).$$

Having identified E_σ^{Hilb} with some subspace of $\widehat{E}_\sigma := \widehat{E} \hat{\otimes}_\sigma \mathbb{C}$, we may also extend the Hilbert norm $\|\cdot\|_\sigma$ on E_σ^{Hilb} to a function

$$\|\cdot\|_\sigma : \widehat{E}_\sigma \longrightarrow [0, +\infty]$$

by letting

$$\|v\|_\sigma := +\infty \text{ for any } v \in \widehat{E}_\sigma \setminus E_\sigma^{\text{Hilb}}.$$

Then the relations

$$\|v\|_\sigma = \sup_{U \in \mathcal{U}(\widehat{E})} \|p_{U,\sigma}(v)\|_{\overline{E}_U,\sigma} = \lim_{U \in \mathcal{U}(\widehat{E})} \|p_{U,\sigma}(v)\|_{\overline{E}_U,\sigma}$$

hold for any $v \in \widehat{E}_\sigma$.

Observe that, for any $R \in \mathbb{R}_+$, the ball $\{v \in E_\sigma^{\text{Hilb}} \mid \|v\| \leq R\}$ is closed in the locally convex \mathbb{C} -vector space \widehat{E}_σ . (Indeed, it is convex and compact in the weak topology of E_σ^{Hilb} , hence in the weak topology of \widehat{E}_σ .)

6.1.5. Direct images. Ind- and pro-euclidean lattices. The construction of the direct image of a hermitian vector bundle over $\text{Spec } \mathcal{O}_K$ by the morphism $\pi : \text{Spec } \mathcal{O}_K \rightarrow \text{Spec } \mathbb{Z}$ discussed in Section above extend to ind- and pro-hermitian vector bundles.

For instance, for any pro-hermitian vector bundle \widehat{E} over $\text{Spec } \mathcal{O}_K$, we may define its direct image by π as the pro-hermitian vector bundle over $\text{Spec } \mathbb{Z}$:

$$\pi_* \widehat{E} := (\pi_* \widehat{E}, (E_{\mathbb{C}}^{\text{Hilb}}, \|\cdot\|_{\mathbb{C}}))$$

where $\pi_* \widehat{E}$ is nothing but \widehat{E} considered as a topological \mathbb{Z} -module, and where $E_{\mathbb{C}}^{\text{Hilb}}$ denotes the complex Hilbert space defined as

$$E_{\mathbb{C}}^{\text{Hilb}} := \bigoplus_{\sigma: K \hookrightarrow \mathbb{C}} E_\sigma^{\text{Hilb}},$$

equipped with the norm $\|\cdot\|_{\mathbb{C}}$ such that, for any $(x_\sigma)_{\sigma: K \hookrightarrow \mathbb{C}} \in E_{\mathbb{C}}^{\text{Hilb}}$,

$$\|(x_\sigma)_{\sigma: K \hookrightarrow \mathbb{C}}\|_{\mathbb{C}}^2 := \sum_{\sigma: K \hookrightarrow \mathbb{C}} \|x_\sigma\|_\sigma^2.$$

(Observe that, since $\widehat{E} \otimes_{\mathbb{Z}} \mathbb{C} \simeq \bigoplus_{\sigma: K \hookrightarrow \mathbb{C}} \widehat{E}_\sigma$, the Hilbert space $E_{\mathbb{C}}^{\text{Hilb}}$ is indeed a dense subspace of $(\pi_* \widehat{E})_{\mathbb{C}}.$)

This construction of direct images reduces many questions concerning ind- and pro-hermitian vector bundles over $\text{Spec } \mathcal{O}_K$ to questions concerning ind- and pro-hermitian vector bundles over the “final” arithmetic curve $\text{Spec } \mathbb{Z}$. We will often say *ind-euclidean lattice* (resp. *pro-euclidean lattice*) instead of ind-hermitian (resp. pro-hermitian) vector bundle over $\text{Spec } \mathbb{Z}$.

Observe also that a pro-hermitian vector bundle $\widehat{E} = (\widehat{E}, (E_{\mathbb{C}}^{\text{Hilb}}, \|\cdot\|_{\mathbb{C}}))$ on $\text{Spec } \mathbb{Z}$ may be equivalently defined as a pair $(\widehat{E}, (E_{\mathbb{R}}, \|\cdot\|_{\mathbb{R}}))$ where $(E_{\mathbb{R}}, \|\cdot\|_{\mathbb{R}})$ denotes a real Hilbert space equipped with a continuous \mathbb{R} -linear injection with dense image

$$E_{\mathbb{R}}^{\text{Hilb}} \hookrightarrow \widehat{E}_{\mathbb{R}} := \widehat{E} \otimes_{\mathbb{Z}} \mathbb{R}.$$

(Indeed the real Hilbert space $(E_{\mathbb{R}}, \|\cdot\|_{\mathbb{R}})$ is deduced from $(E_{\mathbb{C}}, \|\cdot\|_{\mathbb{C}})$ by taking its fixed point under complex conjugation. Conversely, we recover $(E_{\mathbb{C}}, \|\cdot\|_{\mathbb{C}})$ from $(E_{\mathbb{R}}, \|\cdot\|_{\mathbb{R}})$ by extending the scalars from \mathbb{R} to \mathbb{C} .)

Similar remarks concerning ind-hermitian vector bundles might be developed and will be left to the reader. We only observe that an ind-hermitian vector bundle \overline{F} over $\text{Spec } \mathbb{Z}$ may be defined as a pair $(F, \|\cdot\|)$ where F is a countable free \mathbb{Z} -module and $\|\cdot\|$ is a prehilbertian norm on the real vector space $F_{\mathbb{R}} := F \otimes_{\mathbb{Z}} \mathbb{R}$.

6.2. Hilbertisable ind- and pro-vector bundles. In applications, it is convenient to have at one’s disposal weakened variants of the notions of ind- and pro-hermitian vector bundles, where the (pre-)hilbertian norms that enter into their definitions in (6.1.1) and in (6.1.4) are replaced by equivalence classes of (pre-)hilbertian norms.

Thus we shall define a *Hilbertisable ind-vector bundle* over $\text{Spec } \mathcal{O}_K$ as the data

$$\ddot{F} := (F, (F_\sigma^{\text{top}})_{\sigma: \hookrightarrow K})$$

of an object F of $CP_{\mathcal{O}_K}$ and of structures F_σ^{top} of complex topological vector spaces on the \mathbb{C} -vector spaces $F_\sigma := F \otimes_{\mathcal{O}_K} \mathbb{C}$ that may be defined by prehilbertian norms. The conjugations maps $F_\sigma \xrightarrow{\sim} F_{\overline{\sigma}}$ are required to be homeomorphisms.

To any ind-hermitian vector bundle $\overline{F} := (F, (\|\cdot\|_\sigma)_{\sigma: K \hookrightarrow \mathbb{C}})$ is attached its underlying Hilbertisable ind-vector bundle $\check{F} := (F, (F_\sigma^{\text{top}})_{\sigma: K \hookrightarrow \mathbb{C}})$, where F_σ^{top} denotes F_σ equipped with the norm topology defined by $\|\cdot\|_\sigma$.

Similarly, we shall define a *Hilbertisable pro-vector bundle over $\text{Spec } \mathcal{O}_K$* as a pair

$$\tilde{E} := (\hat{E}, (E_\sigma^{\text{Hilb}}, i_\sigma)_{\sigma: K \hookrightarrow \mathbb{C}})$$

where \hat{E} is an object of $CTC_{\mathcal{O}_K}$ and where, for every field embedding $\sigma : K \hookrightarrow \mathbb{C}$, E_σ^{Hilb} denotes a complex Hilbertisable vector space (that is, a topological complex vector space, the topology of which may be defined by some Hilbert space structure) and $i_\sigma : E_\sigma^{\text{Hilb}} \rightarrow \hat{E}_\sigma$ a continuous injective \mathbb{C} -linear map with dense image. These data are required to be compatible with complex conjugation (namely, one requires the existence of \mathbb{C} -antilinear isomorphisms $\gamma_\sigma : E_\sigma^{\text{Hilb}} \xrightarrow{\sim} E_{\bar{\sigma}}^{\text{Hilb}}$ that satisfy the relations (6.3)).

Observe that, by the closed graph theorem, the “Hilbertisable” topology on E_σ^{Hilb} is the unique topology of Fréchet space on the complex vector subspace E_σ^{Hilb} of \hat{E}_σ which makes continuous the injection from E_σ^{Hilb} into \hat{E}_σ .

To any pro-hermitian vector bundle

$$\widehat{E} := (\hat{E}, (\|\cdot\|_\sigma, i_\sigma)_{\sigma: K \hookrightarrow \mathbb{C}})$$

is attached its underlying Hilbertisable pro-vector bundle:

$$\tilde{E} := (\hat{E}, (E_\sigma^{\text{Hilb}}, i_\sigma)_{\sigma: K \hookrightarrow \mathbb{C}}).$$

6.3. Constructions as inductive and projective limits.

6.3.1. *Construction of ind-hermitian vector bundles as inductive limits.* Consider an inductive system

$$\overline{F}_\bullet : \overline{F}_0 \xrightarrow{j_0} \overline{F}_1 \xrightarrow{j_1} \dots \xrightarrow{j_{i-1}} \overline{F}_i \xrightarrow{j_i} \overline{F}_{i+1} \xrightarrow{j_{i+1}} \dots$$

of injective admissible morphisms of hermitian vector bundles over $\text{Spec } \mathcal{O}_K$.

To \overline{F}_\bullet , we may attach an ind-hermitian vector bundle

$$\varinjlim_i \overline{F}_i := (F, (\|\cdot\|_\sigma)_{\sigma: K \hookrightarrow \mathbb{C}})$$

defined by the following simple construction.

Its underlying \mathcal{O}_K -module is the inductive limit

$$F := \varinjlim_i F_i.$$

By construction, it satisfies Condition (4) in Proposition 5.1.1, and therefore is indeed an object of $CP_{\mathcal{O}_K}$.

Moreover, for any field embedding $\sigma : K \hookrightarrow \mathbb{C}$, the maps $j_{i,\sigma} : F_{i,\sigma} \rightarrow F_{i+1,\sigma}$ are isometric with respect to the hermitian norms $\|\cdot\|_{F_{i,\sigma}}$ and $\|\cdot\|_{F_{i+1,\sigma}}$. Therefore there is a unique norm $\|\cdot\|_\sigma$ on $F_\sigma := \varinjlim_i F_{i,\sigma}$ such that the canonical maps $F_{i,\sigma} \hookrightarrow F_\sigma$ are isometric with respect to the norms $\|\cdot\|_{F_{i,\sigma}}$ and $\|\cdot\|_\sigma$. The so-defined norm $\|\cdot\|_\sigma$, like the norms $\|\cdot\|_{F_{i,\sigma}}$, is clearly a prehilbertian norm.

Observe that, up to isometric isomorphism, any ind-hermitian vector bundle

$$\overline{F} := (F, (\|\cdot\|_\sigma)_{\sigma: K \hookrightarrow \mathbb{C}})$$

over $\text{Spec } \mathcal{O}_K$ is the limit of an inductive system \overline{F}_\bullet of hermitian vector bundles as above. Indeed, we may consider a sequence $(F_i)_{i \in \mathbb{N}}$ of \mathcal{O}_K -submodules of F satisfying Condition (4) in Proposition 5.1.1 (with $A = \mathcal{O}_K$), and endow each F_i with the hermitian norms restrictions of the given norms $(\|\cdot\|_\sigma)_{\sigma: K \hookrightarrow \mathbb{C}}$: the so-defined hermitian vector bundles \overline{F}_i define an inductive system \overline{F}_\bullet the limit of which $\varinjlim_i \overline{F}_i$ is canonically isomorphic to \overline{F} .

6.3.2. *Construction of pro-hermitian vector bundles as projective limits.* Consider a projective system

$$\overline{E}_\bullet : \overline{E}_0 \xleftarrow{q_0} \overline{E}_1 \xleftarrow{q_1} \dots \xleftarrow{q_{i-1}} \overline{E}_i \xleftarrow{q_i} \overline{E}_{i+1} \xleftarrow{q_{i+1}} \dots$$

of surjective admissible morphisms of hermitian vector bundles over $\text{Spec } \mathcal{O}_K$.

To \overline{E}_\bullet , we may associate a pro-hermitian vector bundle

$$\varprojlim_i \overline{E}_i = (\widehat{E}, (\overline{E}_U)_{U \in \mathcal{U}(\widehat{E})})$$

over $\text{Spec } \mathcal{O}_K$ defined as follows.

Its underlying topological \mathcal{O}_K -module is the object of $CTC_{\mathcal{O}_K}$ defined as the projective limit

$$\widehat{E} := \varprojlim_i E_i.$$

Let us consider the kernels $U_i := \ker p_i$ of the canonical projections $p_i : \widehat{E} \longrightarrow E_i$. The sequence $(U_i)_{i \in \mathbb{N}}$ is non-increasing and constitutes a basis of neighborhood of 0 in $\mathcal{U}(\widehat{E})$.

For any U in $\mathcal{U}(\widehat{E})$, there exists $i \in \mathbb{N}$ such that U contains U_i . Then the quotient map

$$p_{UU_i} : E_i := \widehat{E}/U_i \longrightarrow E_U := \widehat{E}/U_i$$

is surjective. Consequently, for any embedding $\sigma : K \hookrightarrow \mathbb{C}$, the \mathbb{C} -linear map

$$p_{UU_i, \sigma} : E_{i, \sigma} \longrightarrow E_{U, \sigma}$$

also is surjective, and $E_{U, \sigma}$ may be endowed with the hermitian norm $\|\cdot\|_{E_{U, \sigma}}$ defined as the quotient norm, defined by means of $p_{UU_i, \sigma}$, of the hermitian norm $\|\cdot\|_{E_{i, \sigma}}$ on $E_{i, \sigma}$. By construction, the hermitian vector bundle over $\text{Spec } \mathcal{O}_K$

$$\overline{E}_U := (E_U, (\|\cdot\|_{E_{U, \sigma}})_{\sigma : K \hookrightarrow \mathbb{C}})$$

is such that $p_{UU_i} : E_i \longrightarrow E_U$ becomes a surjective admissible morphism from \overline{E}_i to \overline{E}_U .

Using the fact that the morphisms $q_i : \overline{E}_{i+1} \longrightarrow \overline{E}_i$, and therefore their compositions

$$p_{U_i U_{i'}} = q_i \circ \dots \circ q_{i'-1} : \overline{E}_{i'} \longrightarrow \overline{E}_{i'-1} \longrightarrow \dots \longrightarrow \overline{E}_i,$$

are surjective admissible, one easily checks that the construction of the hermitian structure on \overline{E}_U does not depend on the choice of the open saturated submodule U_i contained in U .

Finally, for any U and U' in $\mathcal{U}(\widehat{E})$ and any $i \in \mathbb{N}$ such that $U_i \subset U \subset U'$, the commutativity of the diagram

$$\begin{array}{ccc} E_i & \xrightarrow{p_{UU_i}} & E_U \\ & \searrow p_{U'U_i} & \downarrow p_{U'U} \\ & & E_{U'} \end{array}$$

and the fact that p_{UU_i} (resp. $p_{U'U_i}$) is a surjective admissible morphism from \overline{E}_i to \overline{E}_U (resp. from \overline{E}_i to $\overline{E}_{U'}$) implies that $p_{U'U}$ is a surjective admissible morphism from \overline{E}_U to $\overline{E}_{U'}$.

Observe that, up to isometric isomorphism, any pro-hermitian vector bundle

$$\widehat{E} := (\widehat{E}, (\overline{E}_U)_{U \in \mathcal{U}(\widehat{E})})$$

over $\text{Spec } \mathcal{O}_K$ is the limit of some projective system of hermitian vector bundle \overline{E}_\bullet as above. Indeed, we may choose a decreasing sequence $(U_i)_{i \in \mathbb{N}}$ in $\mathcal{U}(\widehat{E})$ which constituteS a basis of neighborhoods of 0 in \widehat{E} , and consider the projective system defined by the hermitian vector bundles $\overline{E}_i := \overline{E}_{U_i}$.

6.4. Morphisms between ind- and pro-hermitian vector bundles over \mathcal{O}_K . For any two normed complex vector spaces $(V, \|\cdot\|)$ and $(V', \|\cdot\|')$ and for any λ in \mathbb{R}_+ , we may consider the set of \mathbb{C} -linear maps of operator norm at most λ from $(V, \|\cdot\|)$ to $(V', \|\cdot\|')$:

$$\mathrm{Hom}_{\mathbb{C}}^{\leq \lambda}((V, \|\cdot\|), (V', \|\cdot\|')) := \{T \in \mathrm{Hom}_{\mathbb{C}}(V, V') \mid \|T\| := \sup_{v \in V, \|v\| \leq 1} \|Tv\|' \leq \lambda\}.$$

Their union

$$\mathrm{Hom}_{\mathbb{C}}^{\mathrm{cont}}((V, \|\cdot\|), (V', \|\cdot\|')) := \bigcup_{\lambda \in \mathbb{R}_+} \mathrm{Hom}_{\mathbb{C}}^{\leq \lambda}((V, \|\cdot\|), (V', \|\cdot\|'))$$

is the \mathbb{C} -vector space of continuous linear maps from $(V, \|\cdot\|)$ to $(V', \|\cdot\|')$.

6.4.1. Categories of ind-hermitian vector bundles. Let \overline{F}_1 and \overline{F}_2 be two ind-hermitian vector bundles over \mathcal{O}_K . For any λ in \mathbb{R}_+ , we define $\mathrm{Hom}_{\mathcal{O}_K}^{\leq \lambda}(\overline{F}_2, \overline{F}_1)$ as the subset of $\mathrm{Hom}_{\mathcal{O}_K}(F_2, F_1)$ consisting of the \mathcal{O}_K -linear maps

$$\psi : F_2 \longrightarrow F_1$$

such that, for every embedding $\sigma : K \hookrightarrow \mathbb{C}$, the induced \mathbb{C} -linear map

$$\psi_{\sigma} : F_{2,\sigma} \longrightarrow F_{1,\sigma}$$

is continuous, of operator norm $\leq \lambda$, when $F_{2,\sigma}$ and $F_{1,\sigma}$ are equipped with the pre-hilbertian norms $\|\cdot\|_{\overline{F}_{2,\sigma}}$ and $\|\cdot\|_{\overline{F}_{1,\sigma}}$.

Clearly, if \overline{F}_1 , \overline{F}_2 , and \overline{F}_3 are ind-hermitian vector bundles over \mathcal{O}_K , and if λ and μ are two elements of \mathbb{R}_+ , the composition of an element ψ in $\mathrm{Hom}_{\mathcal{O}_K}^{\leq \lambda}(\overline{F}_2, \overline{F}_1)$ and of an element ψ' in $\mathrm{Hom}_{\mathcal{O}_K}^{\leq \mu}(\overline{F}_3, \overline{F}_2)$ defines an element $\psi \circ \psi'$ in $\mathrm{Hom}_{\mathcal{O}_K}^{\leq \lambda + \mu}(\overline{F}_3, \overline{F}_1)$.

Consequently, it is possible to define a category whose objects are the ind-hermitian vector bundles over \mathcal{O}_K by either of the following constructions:

(a) by defining the morphisms from \overline{F}_2 to \overline{F}_1 to be

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_K}^{\mathrm{cont}}(\overline{F}_2, \overline{F}_1) &:= \bigcup_{\lambda \in \mathbb{R}_+} \mathrm{Hom}_{\mathcal{O}_K}^{\leq \lambda}(\overline{F}_2, \overline{F}_1) \\ &= \left\{ \psi \in \mathrm{Hom}_{\mathcal{O}_K}(F_2, F_1) \mid \text{for every } \sigma : K \hookrightarrow \mathbb{C}, \psi_{\sigma} \in \mathrm{Hom}_{\mathbb{C}}^{\mathrm{cont}}((F_{2,\sigma}, \|\cdot\|_{\overline{F}_{2,\sigma}}), (F_{1,\sigma}, \|\cdot\|_{\overline{F}_{1,\sigma}})) \right\}. \end{aligned}$$

The so-defined category $\mathrm{ind}\overline{\mathrm{Vect}}^{\mathrm{cont}}(\mathcal{O}_K)$ is clearly \mathcal{O}_K -linear.

(b) by defining the morphisms from \overline{F}_2 to \overline{F}_1 to be $\mathrm{Hom}_{\mathcal{O}_K}^{\leq 1}(\overline{F}_2, \overline{F}_1)$. The so-defined category will be denoted by $\mathrm{ind}\overline{\mathrm{Vect}}^{\leq 1}(\mathcal{O}_K)$.

Observe that an isomorphism $\psi : \overline{F}_2 \xrightarrow{\sim} \overline{F}_1$ in $\mathrm{ind}\overline{\mathrm{Vect}}^{\leq 1}(\mathcal{O}_K)$ (resp. in $\mathrm{ind}\overline{\mathrm{Vect}}^{\mathrm{cont}}(\mathcal{O}_K)$) is an isomorphism of \mathcal{O}_K -modules $\psi : F_2 \xrightarrow{\sim} F_1$ such that the \mathbb{C} -linear isomorphisms $\psi_{\sigma} : F_{2,\sigma} \xrightarrow{\sim} F_{1,\sigma}$ is an isometry (resp. a homeomorphism) between the normed vector spaces $(F_{2,\sigma}, \|\cdot\|_{\overline{F}_{2,\sigma}})$ and $(F_{1,\sigma}, \|\cdot\|_{\overline{F}_{1,\sigma}})$. In particular, isometric isomorphisms of ind-hermitian vector bundles (as defined in paragraph 6.1.1) are exactly the isomorphisms in $\mathrm{ind}\overline{\mathrm{Vect}}^{\leq 1}(\mathcal{O}_K)$.

The inductive limit $\varinjlim_i \overline{F}_i$ of an inductive system \overline{F}_{\bullet} of injective admissible morphisms of hermitian vector bundles, as considered in paragraph 6.3.1, together with the obvious inclusion maps $\overline{F}_k \longrightarrow \varinjlim_i \overline{F}_i$, is easily checked to be a inductive limit of \overline{F}_{\bullet} in the category $\mathrm{ind}\overline{\mathrm{Vect}}^{\leq 1}(\mathcal{O}_K)$.

We may also introduce an \mathcal{O}_K -linear category $\widetilde{\mathrm{ind}\overline{\mathrm{Vect}}}(\mathcal{O}_K)$, whose objects are the Hilbertisable ind-vector bundles over $\mathrm{Spec} \mathcal{O}_K$: in the category $\widetilde{\mathrm{ind}\overline{\mathrm{Vect}}}(\mathcal{O}_K)$, the set of morphisms between two

Hilbertisable ind-vector bundles $\ddot{F}_2 := (F_2, (F_{2,\sigma}^{\text{top}})_{\sigma: \hookrightarrow K})$ and $\ddot{F}_1 := (F_1, (F_{1,\sigma}^{\text{top}})_{\sigma: \hookrightarrow K})$ over $\text{Spec } \mathcal{O}_K$ is defined as the \mathcal{O}_K -module

$$\begin{aligned} \text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\ddot{F}_2, \ddot{F}_1) \\ = \{ \psi \in \text{Hom}_{\mathcal{O}_K}(F_2, F_1) \mid \text{for every } \sigma : K \hookrightarrow \mathbb{C}, \psi_\sigma : F_{2,\sigma}^{\text{top}} \longrightarrow F_{1,\sigma}^{\text{top}} \text{ is continuous} \}. \end{aligned}$$

Finally there is a natural forgetful functor from $\text{ind}\widetilde{\text{Vect}}^{\text{cont}}(\mathcal{O}_K)$ to $\text{ind}\widetilde{\text{Vect}}(\mathcal{O}_K)$, which maps an ind-hermitian vector bundle \overline{F} to the associated prehilbertisable ind-vector bundle \ddot{F} , and is the identity on morphisms. It is easily seen to be an equivalence of category.

6.4.2. Categories of pro-hermitian vector bundles. For any two pro-hermitian vector bundles \widehat{E}_1 and \widehat{E}_2 over $\text{Spec } \mathcal{O}_K$ and for any λ in \mathbb{R}_+ , we define:

$$(6.4) \quad \text{Hom}_{\mathcal{O}_K}^{\leq \lambda}(\widehat{E}_1, \widehat{E}_2) := \varinjlim_{U_2} \varinjlim_{U_1} \text{Hom}_{\mathcal{O}_K}^{\leq \lambda}(\overline{E}_{1,U_1}, \overline{E}_{2,U_2}).$$

(In the inductive (resp. projective) limit, U_1 (resp. U_2) varies in the filtered set $\mathcal{U}(\widehat{E}_1)$, ordered by \supseteq .)

We also define the set of *continuous \mathcal{O}_K -morphisms from \widehat{E}_1 and \widehat{E}_2* as the \mathcal{O}_K -module

$$(6.5) \quad \text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\widehat{E}_1, \widehat{E}_2) := \bigcup_{\lambda \in \mathbb{R}_+} \text{Hom}_{\mathcal{O}_K}^{\leq \lambda}(\widehat{E}_1, \widehat{E}_2).$$

Observe that an element $\tilde{\phi}$ in $\text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\widehat{E}_1, \widehat{E}_2)$ is uniquely determined by its image $\hat{\phi}$ in the \mathcal{O}_K -module

$$\varinjlim_{U_2} \varinjlim_{U_1} \text{Hom}_{\mathcal{O}_K}(E_{1,U_1}, E_{2,U_2}) \simeq \text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\widehat{E}_1, \widehat{E}_2)$$

of \mathcal{O}_K -linear continuous maps from \widehat{E}_1 to \widehat{E}_2 .

The following proposition is a direct consequence of the construction of the Hilbert spaces associated to a pro-hermitian vector bundle:

Proposition 6.4.1. *With the above notation, an element $\hat{\phi}$ in $\text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\widehat{E}_1, \widehat{E}_2)$ may be lifted to an element $\tilde{\phi}$ in $\text{Hom}_{\mathcal{O}_K}^{\leq \lambda}(\widehat{E}_1, \widehat{E}_2)$ if and only if, for every embedding $\sigma : K \hookrightarrow \mathbb{C}$, there exists a continuous \mathbb{C} -linear map of operator norm $\leq \lambda$*

$$\phi_\sigma : E_{1,\sigma}^{\text{Hilb}} \longrightarrow E_{2,\sigma}^{\text{Hilb}}$$

between the Hilbert spaces $\overline{E}_{1,\sigma}^{\text{Hilb}}$ and $\overline{E}_{2,\sigma}^{\text{Hilb}}$ such that the following diagram is commutative:

$$(6.6) \quad \begin{array}{ccc} E_{1,\sigma}^{\text{Hilb}} & \xrightarrow{\phi_\sigma} & E_{2,\sigma}^{\text{Hilb}} \\ i_{\sigma}^{\widehat{E}_1} \downarrow & & \downarrow i_{\sigma}^{\widehat{E}_2} \\ \widehat{E}_{1,\sigma} & \xrightarrow{\hat{\phi}_\sigma} & \widehat{E}_{2,\sigma}. \end{array}$$

When this holds, these morphisms ϕ_σ are unique. □

According to Proposition 6.4.1, an element of $\text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\widehat{E}_1, \widehat{E}_2)$ (resp. $\text{Hom}_{\mathcal{O}_K}^{\leq \lambda}(\widehat{E}_1, \widehat{E}_2)$) may be described as a pair

$$(6.7) \quad \tilde{\phi} := (\hat{\phi}, (\phi_\sigma)_{\sigma: K \hookrightarrow \mathbb{C}}),$$

consisting in the following data:

- (1) a continuous morphism of topological \mathcal{O}_K -modules

$$\hat{\phi} : \hat{E}_1 \longrightarrow \hat{E}_2;$$

- (2) for every embedding $\sigma : K \hookrightarrow \mathbb{C}$, a continuous \mathbb{C} -linear map continuous map (resp. of operator norm $\leq \lambda$)

$$\phi_\sigma : E_{1,\sigma}^{\text{Hilb}} \longrightarrow E_{2,\sigma}^{\text{Hilb}}$$

between the Hilbert spaces $\overline{E}_{1,\sigma}^{\text{Hilb}}$ and $\overline{E}_{2,\sigma}^{\text{Hilb}}$ that is compatible with $\hat{\phi}$, in the sense that the diagram (6.6) is commutative for every embedding $\sigma : K \hookrightarrow \mathbb{C}$.

In the following sections, and in the sequel of this article, we shall freely use this alternative description of $\text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\hat{E}_1, \hat{E}_2)$.

As a special case of this description, observe that, for any hermitian vector bundle \overline{E} over $\text{Spec } \mathcal{O}_K$ and for any pro-hermitian vector bundle \hat{F} over $\text{Spec } \mathcal{O}_K$, we have a natural identification:

$$\text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\overline{E}, \hat{F}) \xrightarrow{\sim} \text{Hom}_{\mathcal{O}_K}(E, \hat{F} \cap F_{\mathbb{C}}^{\text{Hilb}}).$$

For any three pro-hermitian vector bundles \hat{E}_1 , \hat{E}_2 , and \hat{E}_3 over $\text{Spec } \mathcal{O}_K$, there is a natural \mathcal{O}_K -bilinear composition map

$$(6.8) \quad \cdot \circ \cdot : \text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\hat{E}_2, \hat{E}_3) \times \text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\hat{E}_1, \hat{E}_2) \longrightarrow \text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\hat{E}_1, \hat{E}_3),$$

— that, for any $(\lambda, \mu) \in \mathbb{R}_+^2$, maps $\text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\hat{E}_2, \hat{E}_3) \times \text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\hat{E}_1, \hat{E}_2)$ to $\text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\hat{E}_1, \hat{E}_3)$ — deduced from the composition of morphisms

$$\text{Hom}_{\mathcal{O}_K}^{\leq \lambda}(\hat{E}_{2,U_2}, \hat{E}_{3,U_3}) \times \text{Hom}_{\mathcal{O}_K}^{\leq \mu}(\hat{E}_{1,U_1}, \hat{E}_{2,U_2}) \longrightarrow \text{Hom}_{\mathcal{O}_K}^{\leq \lambda\mu}(\hat{E}_{1,U_1}, \hat{E}_{3,U_3})$$

by passage to the projective and inductive limits involved in the definition (see (6.4)) of continuous \mathcal{O}_K -morphisms of pro-hermitian vector bundles.

In terms of the description of the morphisms of pro-hermitian vector bundles as pairs of the form (6.7), this composition law may be described as follows. If $\tilde{\phi} := (\hat{\phi}, (\phi_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}})$ (resp. $\tilde{\psi} := (\hat{\psi}, (\psi_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}})$) is an element of $\text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\hat{E}_1, \hat{E}_2)$ (resp. of $\text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\hat{E}_2, \hat{E}_3)$), the composition law (6.8) maps $(\tilde{\psi}, \tilde{\phi})$ to

$$\tilde{\psi} \circ \tilde{\phi} := (\hat{\psi} \circ \hat{\phi}, (\psi_\sigma \circ \phi_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}}).$$

By defining the morphisms from \hat{E}_1 to \hat{E}_2 to be $\text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\hat{E}_1, \hat{E}_2)$ and the composition of morphisms as above, the class of pro-hermitian vector bundles over $\text{Spec } \mathcal{O}_K$ becomes an \mathcal{O}_K -linear category. We shall denote it by $\text{proVect}^{\text{cont}}(\mathcal{O}_K)$.

We may define another category, the objects of which are again the pro-hermitian vector bundles over $\text{Spec } \mathcal{O}_K$, by defining the morphisms the morphisms from \hat{E}_1 to \hat{E}_2 to be $\text{Hom}_{\mathcal{O}_K}^{\leq 1}(\hat{E}_1, \hat{E}_2)$, and by defining the composition as above. We shall denote this category by $\text{proVect}^{\leq 1}(\mathcal{O}_K)$.

Finally we may formulate some observations concerning these categories of pro-hermitian vector bundles similar to the ones concerning ind-hermitian vector bundles at the end of paragraph 6.4.1:

- (i) An isomorphism $\tilde{\phi} : \hat{E}_1 \xrightarrow{\sim} \hat{E}_2$ in $\text{proVect}^{\text{cont}}(\mathcal{O}_K)$ (resp. in $\text{proVect}^{\leq 1}(\mathcal{O}_K)$) is the data of an isomorphism $\hat{\phi} : \hat{E}_1 \xrightarrow{\sim} \hat{E}_2$ of topological \mathcal{O}_K -modules and of \mathbb{C} -linear homeomorphisms $\phi_\sigma : E_{1,\sigma}^{\text{Hilb}} \xrightarrow{\sim} E_{2,\sigma}^{\text{Hilb}}$ compatibel with $\hat{\phi}$.

An isomorphism $\tilde{\phi} : \hat{E}_1 \xrightarrow{\sim} \hat{E}_2$ in $\text{proVect}^{\leq 1}(\mathcal{O}_K)$ is precisely an isometric isomorphism of pro-hermitian vector bundles.

(ii) The projective limit $\varprojlim_i \overline{E}_i$ of a projective system \overline{E}_\bullet of surjective admissible morphisms of hermitian vector bundles (as considered in paragraph 6.3.2), equipped with the projection maps $\varprojlim_i \overline{E}_i \longrightarrow \overline{E}_k$, is easily checked to be a projective limit of \overline{E}_\bullet in $\text{pro}\overline{\text{Vect}}^{\leq 1}(\mathcal{O}_K)$.

(iii) We may introduce the \mathcal{O}_K -linear category $\text{pro}\widetilde{\text{Vect}}(\mathcal{O}_K)$, whose objects are the Hilbertisable pro-vector bundles over $\text{Spec } \mathcal{O}_K$, and a forgetful functor

$$\text{pro}\overline{\text{Vect}}^{\text{cont}}(\mathcal{O}_K) \longrightarrow \text{pro}\widetilde{\text{Vect}}(\mathcal{O}_K)$$

— it sends an object $\widehat{\overline{E}}$ of $\text{pro}\overline{\text{Vect}}^{\text{cont}}(\mathcal{O}_K)$ to the underlying object \tilde{E} of $\text{pro}\widetilde{\text{Vect}}(\mathcal{O}_K)$ — that is actually an equivalence of category.

6.5. The duality between ind- and pro-hermitian vector bundles. In this Section, we construct a duality between the categories $\text{ind}\overline{\text{Vect}}^{\leq 1}(\mathcal{O}_K)$ and $\text{pro}\overline{\text{Vect}}^{\leq 1}(\mathcal{O}_K)$ and between the categories $\text{pro}\overline{\text{Vect}}^{\text{cont}}(\mathcal{O}_K)$ and $\text{pro}\widetilde{\text{Vect}}^{\text{cont}}(\mathcal{O}_K)$ by combining the duality between $CP_{\mathcal{O}_K}$ and $CTC_{\mathcal{O}_K}$ described in Section 5.3 and the classical duality theory of (pre-)Hilbert spaces.

6.5.1. The duality functors.

(i) Let $\overline{F} := (F, (\|\cdot\|_\sigma)_{\sigma: K \hookrightarrow \mathbb{C}})$ be an ind-hermitian vector bundle over $\text{Spec } \mathcal{O}_K$. To \overline{F} , we may attach a dual pro-hermitian vector bundle

$$(6.9) \quad \overline{F}^\vee := (F^\vee, (F_\sigma^{\vee \text{Hilb}}, \|\cdot\|_\sigma^\vee)_{\sigma: K \hookrightarrow \mathbb{C}})$$

over $\text{Spec } \mathcal{O}_K$, defined as follows.

Its underlying topological \mathcal{O}_K -module is the \mathcal{O}_K -module

$$F^\vee := \text{Hom}_{\mathcal{O}_K}(F, \mathcal{O}_K)$$

equipped with the topology of pointwise convergence. In other words, it is the dual in $CTC_{\mathcal{O}_K}$ of the object F of $CP_{\mathcal{O}_K}$.

For any embedding $\sigma : K \hookrightarrow \mathbb{C}$, we get canonical isomorphisms of complex vector spaces

$$(6.10) \quad F_\sigma^\vee := F^\vee \hat{\otimes}_{\mathcal{O}_K, \sigma} \mathbb{C} \simeq \text{Hom}_{\mathcal{O}_K, \sigma}(F, \mathbb{C}) \simeq \text{Hom}_{\mathbb{C}}(F_\sigma, \mathbb{C}).$$

Actually the Fréchet topology on F_σ^\vee (deduced from the structure of topological \mathcal{O}_K -module on F^\vee , as explained in 6.1.3 (i)) coincides with the locally convex topology on $\text{Hom}_{\mathbb{C}}(F_\sigma, \mathbb{C})$ defined by the pointwise convergence on F_σ .

By means of the identifications (6.10), we may introduce the vector subspace

$$F_\sigma^{\vee \text{Hilb}} := \text{Hom}_{\mathbb{C}}^{\text{cont}}((F_\sigma, \|\cdot\|_\sigma), \mathbb{C})$$

of F_σ^\vee consisting in the linear forms on F_σ continuous with respect to the hermitian norm $\|\cdot\|_\sigma$. Equipped with its operator norm, defined by

$$\|\xi\|_\sigma^\vee := \sup_{f \in F_\sigma, \|f\|_\sigma \leq 1} |\xi(f)|,$$

this vector space becomes a separable Hilbert space $(F_\sigma^{\vee \text{Hilb}}, \|\cdot\|_\sigma^\vee)$, and the inclusion

$$F_\sigma^{\vee \text{Hilb}} \hookrightarrow F_\sigma^\vee$$

is easily seen to be continuous and to have a dense image.²⁶

This construction is compatible with complex conjugation, and the right hand of (6.9) actually defines a pro-hermitian vector bundle over $\text{Spec } \mathcal{O}_K$, in terms of the alternative approach to pro-hermitian vector bundles presented in paragraph 6.1.4.

²⁶Actually, from any countable \mathbb{C} -basis of F_σ , by orthonormalization we get a countable orthonormal basis of F_σ . By means of any such orthonormal basis, we get compatible isomorphisms: $F_\sigma \xrightarrow{\sim} \mathbb{C}^{(\mathbb{N})}$, $F_\sigma^\vee \xrightarrow{\sim} \mathbb{C}^{\mathbb{N}}$, and $F_\sigma^{\vee \text{Hilb}} \xrightarrow{\sim} \ell^2(\mathbb{N})$.

Let λ be a positive real number, and let $\overline{F}_1 := (F_1, (\|\cdot\|_{1,\sigma})_{\sigma:K \hookrightarrow \mathbb{C}})$ and $\overline{F}_2 := (F_2, (\|\cdot\|_{2,\sigma})_{\sigma:K \hookrightarrow \mathbb{C}})$ be two ind-hermitian vector bundles over $\text{Spec } \mathcal{O}_K$. Consider a morphism α in $\text{Hom}_{\mathcal{O}_K}^{\leq \lambda}(\overline{F}_1, \overline{F}_2)$. By definition, α is an element of $\text{Hom}_{\mathcal{O}_K}(F_1, F_2)$ such that, for any $\sigma : K \hookrightarrow \mathbb{C}$, the \mathbb{C} -linear map $\alpha_\sigma : F_{1,\sigma} \rightarrow F_{2,\sigma}$ is continuous of operator norm $\leq \lambda$ from $(F_{1,\sigma}, \|\cdot\|_{1,\sigma})$ to $(F_{2,\sigma}, \|\cdot\|_{2,\sigma})$.

The dual (or adjoint) morphism

$$\alpha^\vee := . \circ \alpha : F_2^\vee := \text{Hom}_{\mathcal{O}_K}(F_2, \mathcal{O}_K) \rightarrow F_1^\vee := \text{Hom}_{\mathcal{O}_K}(F_1, \mathcal{O}_K)$$

becomes, after “completed base change” under the embedding $\sigma : \mathcal{O}_K \hookrightarrow \mathbb{C}$, the \mathbb{C} -linear map

$$\alpha_\sigma^\vee := . \circ \alpha_\sigma : F_{2,\sigma}^\vee \simeq \text{Hom}_{\mathbb{C}}(F_{2,\sigma}, \mathbb{C}) \rightarrow F_{1,\sigma}^\vee \simeq \text{Hom}_{\mathbb{C}}(F_{1,\sigma}, \mathbb{C}).$$

It is a continuous \mathbb{C} -linear map between the Fréchet spaces $F_{2,\sigma}^\vee$ and $F_{1,\sigma}^\vee$. Moreover, it sends $F_{2,\sigma}^{\vee\text{Hilb}} := \text{Hom}_{\mathbb{C}}^{\text{cont}}((F_{2,\sigma}, \|\cdot\|_{2,\sigma}), \mathbb{C})$ to $F_{1,\sigma}^{\vee\text{Hilb}} := \text{Hom}_{\mathbb{C}}^{\text{cont}}((F_{1,\sigma}, \|\cdot\|_{1,\sigma}), \mathbb{C})$, and defines a continuous \mathbb{C} -linear map of operator norm $\leq \lambda$ from $(F_{2,\sigma}^{\vee\text{Hilb}}, \|\cdot\|_{2,\sigma}^\vee)$ to $(F_{1,\sigma}^{\vee\text{Hilb}}, \|\cdot\|_{1,\sigma}^\vee)$.

In conclusion, the dual map α^\vee defines a morphism in $\text{Hom}_{\mathcal{O}_K}^{\leq \lambda}(\overline{F}_2^\vee, \overline{F}_1^\vee)$. Moreover, the so-defined maps

$$\begin{aligned} \text{Hom}_{\mathcal{O}_K}^{\leq \lambda}(\overline{F}_1, \overline{F}_2) &\rightarrow \text{Hom}_{\mathcal{O}_K}^{\leq \lambda}(\overline{F}_2^\vee, \overline{F}_1^\vee) \\ \alpha &\mapsto \alpha^\vee. \end{aligned}$$

define an \mathcal{O}_K -linear map

$$.^\vee : \text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\overline{F}_1, \overline{F}_2) \rightarrow \text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\overline{F}_2^\vee, \overline{F}_1^\vee).$$

This construction is clearly functorial and defines two contravariant duality functors,

$$.^\vee : \text{indVect}^{\leq 1}(\mathcal{O}_K) \rightarrow \text{proVect}^{\leq 1}(\mathcal{O}_K)$$

and

$$.^\vee : \text{indVect}^{\text{cont}}(\mathcal{O}_K) \rightarrow \text{proVect}^{\text{cont}}(\mathcal{O}_K),$$

the second of which is \mathcal{O}_K -linear.

Observe that any inductive system

$$\overline{F}_\bullet : \overline{F}_0 \xrightarrow{j_0} \overline{F}_1 \xrightarrow{j_1} \dots \xrightarrow{j_{i-1}} \overline{F}_i \xrightarrow{j_i} \overline{F}_{i+1} \xrightarrow{j_{i+1}} \dots$$

of injective admissible morphisms of hermitian vector bundles over $\text{Spec } \mathcal{O}_K$ determines by duality a projective system

$$\overline{F}_\bullet^\vee : \overline{F}_0^\vee \xleftarrow{j_0^\vee} \overline{F}_1^\vee \xleftarrow{j_1^\vee} \dots \xleftarrow{j_{i-1}^\vee} \overline{F}_i^\vee \xleftarrow{j_i^\vee} \overline{F}_{i+1}^\vee \xleftarrow{j_{i+1}^\vee} \dots$$

of surjective admissible morphisms of hermitian vector bundles over $\text{Spec } \mathcal{O}_K$. The injection morphisms $j_k : \overline{F}_k \rightarrow \varinjlim_i \overline{F}_i$ define, by duality, morphisms in $\text{proVect}^{\leq 1}(\mathcal{O}_K)$:

$$j_k^\vee : (\varinjlim_i \overline{F}_i)^\vee \rightarrow \overline{F}_k^\vee,$$

which in turn define a morphism from $(\varinjlim_i \overline{F}_i)^\vee$ to the projective limit $\varprojlim_i \overline{F}_i^\vee$ of $\overline{F}_\bullet^\vee$, that is easily seen to be an isometric isomorphism of pro-hermitian vector bundles over $\text{Spec } \mathcal{O}_K$:

$$(6.11) \quad (\varinjlim_i \overline{F}_i)^\vee \xrightarrow{\sim} \varprojlim_i \overline{F}_i^\vee.$$

(ii) Conversely, to any pro-hermitian vector bundle over $\text{Spec } \mathcal{O}_K$,

$$\widehat{E} := (\widehat{E}, (E_\sigma^{\text{Hilb}}, \|\cdot\|_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}}),$$

we may attach a dual ind-hermitian vector bundle

$$(6.12) \quad \widehat{E}^\vee := (\widehat{E}^\vee, (\|\cdot\|_\sigma^\vee)_{\sigma:K \hookrightarrow \mathbb{C}})$$

defined as follows.

Its underlying projective \mathcal{O}_K -module is the dual

$$\widehat{E}^\vee := \text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\widehat{E}, \mathcal{O}_K)$$

in $CP_{\mathcal{O}_K}$ of the object \widehat{E} of $CTC_{\mathcal{O}_K}$. Actually, as a consequence of Proposition 5.2.4, \widehat{E}^\vee coincides with the algebraic dual $\text{Hom}_{\mathcal{O}_K}(\widehat{E}, \mathcal{O}_K)$ of the \mathcal{O}_K -module \widehat{E} .

For any embedding $\sigma : K \hookrightarrow \mathbb{C}$, we have canonical isomorphisms:

$$(\widehat{E}^\vee)_\sigma := \widehat{E}^\vee \otimes_{\mathcal{O}_K, \sigma} \mathbb{C} \simeq \left(\lim_{U \in \mathcal{U}(\widehat{E})} E_U^\vee \right) \otimes_{\mathcal{O}_K, \sigma} \mathbb{C} \simeq \lim_{U \in \mathcal{U}(\widehat{E})} E_{U, \sigma}^\vee \simeq \text{Hom}_{\mathbb{C}}^{\text{cont}}(\widehat{E}_\sigma, \mathbb{C}).$$

Since the injection $i_\sigma : E_\sigma^{\text{Hilb}} \hookrightarrow \widehat{E}_\sigma$ is continuous with dense image, its transpose defines an injective map (also with dense image)

$${}^t i_\sigma : (\widehat{E}^\vee)_\sigma \hookrightarrow (E_\sigma^{\text{Hilb}})^\vee$$

from $(\widehat{E}^\vee)_\sigma$ to the dual Hilbert space $(E_\sigma^{\text{Hilb}})^\vee$, and we define the norm $\|\cdot\|_\sigma^\vee$ as the restriction to $(\widehat{E}^\vee)_\sigma$ of the Hilbert norm on $(\widehat{E}^\vee)_\sigma$ dual of the norm $\|\cdot\|_\sigma$ on E_σ^{Hilb} . (In other words, the norm $\|\xi\|_\sigma^\vee$ of some linear form ξ in $\text{Hom}_{\mathbb{C}}^{\text{cont}}(\widehat{E}_\sigma, \mathbb{C})$ is the operator norm, with respect to $\|\cdot\|_\sigma$, of its restriction to E_σ^{Hilb} .)

It is straightforward that the construction of the norms $(\|\cdot\|_\sigma^\vee)_{\sigma: K \hookrightarrow \mathbb{C}}$ on the complex vector spaces $((\widehat{E}^\vee)_\sigma)_{\sigma: K \hookrightarrow \mathbb{C}}$ is compatible with complex conjugation. Consequently the right hand-side of (6.12) indeed defines some ind-hermitian vector bundle over $\text{Spec } \mathcal{O}_K$.

Let λ be a positive real number, and let

$$\widehat{E}_1 := (\widehat{E}_1, (E_{1, \sigma}^{\text{Hilb}}, \|\cdot\|_{1, \sigma})_{\sigma: K \hookrightarrow \mathbb{C}})$$

and

$$\widehat{E}_2 := (\widehat{E}_2, (E_{2, \sigma}^{\text{Hilb}}, \|\cdot\|_{2, \sigma})_{\sigma: K \hookrightarrow \mathbb{C}})$$

be two pro-hermitian vector bundles over $\text{Spec } \mathcal{O}_K$. Consider a morphism β in $\text{Hom}_{\mathcal{O}_K}^{\leq \lambda}(\widehat{E}_1, \widehat{E}_2)$. By definition, β is an element in $\text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\widehat{E}_1, \widehat{E}_2)$ such that, for any embedding $\sigma : K \hookrightarrow \mathbb{C}$, the continuous linear map of Fréchet spaces

$$(6.13) \quad \beta_\sigma : \widehat{E}_{1, \sigma} \longrightarrow \widehat{E}_{2, \sigma}$$

maps $E_{1, \sigma}^{\text{Hilb}}$ to $E_{2, \sigma}^{\text{Hilb}}$, with an operator norm (with respect to the Hilbert norms $\|\cdot\|_{1, \sigma}$ and $\|\cdot\|_{2, \sigma}$) at most λ .

The dual morphism

$$\beta^\vee : \cdot \circ \beta : \widehat{E}_2^\vee := \text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\widehat{E}_2, \mathcal{O}_K) \longrightarrow \widehat{E}_1^\vee := \text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\widehat{E}_1, \mathcal{O}_K),$$

after the base change $\sigma : \mathcal{O}_K \hookrightarrow \mathbb{C}$, becomes the transpose of the map (6.13), namely

$$\beta_\sigma^\vee = \cdot \circ \beta_\sigma : (\widehat{E}_2^\vee)_\sigma \simeq \text{Hom}_{\mathbb{C}}^{\text{cont}}(\widehat{E}_{2, \sigma}, \mathbb{C}) \longrightarrow (\widehat{E}_1^\vee)_\sigma \simeq \text{Hom}_{\mathbb{C}}^{\text{cont}}(\widehat{E}_{1, \sigma}, \mathbb{C}),$$

and therefore satisfies, for any $\xi \in (\widehat{E}_2^\vee)_\sigma$:

$$\|\beta_\sigma^\vee(\xi)\|_{1, \sigma}^\vee = \|\xi \circ \beta_\sigma\|_{1, \sigma}^\vee \leq \lambda \|\xi\|_{2, \sigma}^\vee.$$

This shows that β^\vee belongs to $\text{Hom}_{\mathcal{O}_K}^{\leq \lambda}(\widehat{E}_2^\vee, \widehat{E}_1^\vee)$.

The so-defined maps

$$\begin{array}{ccc} \text{Hom}_{\mathcal{O}_K}^{\leq \lambda}(\widehat{E}_1, \widehat{E}_2) & \longrightarrow & \text{Hom}_{\mathcal{O}_K}^{\leq \lambda}(\widehat{E}_2^\vee, \widehat{E}_1^\vee) \\ \beta & \longmapsto & \beta^\vee. \end{array}$$

define an \mathcal{O}_K -linear map

$$\cdot^\vee : \text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\widehat{E}_1, \widehat{E}_2) \longrightarrow \text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\widehat{E}_2^\vee, \widehat{E}_1^\vee).$$

This construction is clearly functorial and defines two contravariant duality functors,

$$(6.14) \quad {}^\vee : \text{pro}\overline{\text{Vect}}^{\leq 1}(\mathcal{O}_K) \longrightarrow \text{ind}\overline{\text{Vect}}^{\leq 1}(\mathcal{O}_K)$$

and

$${}^\vee : \text{pro}\overline{\text{Vect}}^{\text{cont}}(\mathcal{O}_K) \longrightarrow \text{ind}\overline{\text{Vect}}^{\text{cont}}(\mathcal{O}_K),$$

the second of which is \mathcal{O}_K -linear.

The duality functor (6.14) is compatible with the construction of pro-hermitian vector bundles as limits of projective systems of surjective admissible maps of hermitian vector bundles over $\text{Spec } \mathcal{O}_K$ discussed in paragraph 6.3.2. Namely, if

$$\overline{E}_\bullet : \overline{E}_0 \xleftarrow{q_0} \overline{E}_1 \xleftarrow{q_1} \dots \xleftarrow{q_{i-1}} \overline{E}_i \xleftarrow{q_i} \overline{E}_{i+1} \xleftarrow{q_{i+1}} \dots$$

is such a projective system, we may form the dual inductive system

$$\overline{E}_\bullet^\vee : \overline{E}_0^\vee \xrightarrow{q_0^\vee} \overline{E}_1^\vee \xrightarrow{q_1^\vee} \dots \xrightarrow{q_{i-1}^\vee} \overline{E}_i^\vee \xrightarrow{q_i^\vee} \overline{E}_{i+1}^\vee \xrightarrow{q_{i+1}^\vee} \dots$$

of injective admissible morphisms of hermitian vector bundles over $\text{Spec } \mathcal{O}_K$. The projections $p_k : \varprojlim_i \overline{E}_i \longrightarrow \overline{E}_k$ define, by duality, morphisms in $\text{ind}\overline{\text{Vect}}^{\leq 1}(\mathcal{O}_K)$,

$$p_k^\vee : \overline{E}_k^\vee \longrightarrow (\varprojlim_i \overline{E}_i)^\vee,$$

that in turn define a morphism from the inductive limit of $\overline{E}_\bullet^\vee$ to $(\varprojlim_i \overline{E}_i)^\vee$. This morphism is easily seen to be an isometric isomorphism of ind-hermitian vector bundles over $\text{Spec } \mathcal{O}_K$:

$$(6.15) \quad \varinjlim_i \overline{E}_i^\vee \xrightarrow{\sim} (\varprojlim_i \overline{E}_i)^\vee.$$

6.5.2. Duality as adjoint equivalences. Let $\widehat{E} := (\widehat{E}, (E_\sigma^{\text{Hilb}}, \|\cdot\|_{E,\sigma})_{\sigma:K \hookrightarrow \mathbb{C}})$ be an object of the category $\text{pro}\overline{\text{Vect}}(\mathcal{O}_K)$ and let $\overline{F} := (F, (\|\cdot\|_{F,\sigma})_{\sigma:K \hookrightarrow \mathbb{C}})$ be an object of $\text{ind}\overline{\text{Vect}}(\mathcal{O}_K)$. We may consider their dual objects $\widehat{E}^\vee = (\widehat{E}^\vee, (\|\cdot\|_{\widehat{E}^\vee,\sigma})_{\sigma:K \hookrightarrow \mathbb{C}})$ and $\overline{F}^\vee = (F^\vee, (\|\cdot\|_{F^\vee,\sigma})_{\sigma:K \hookrightarrow \mathbb{C}})$ in $\text{ind}\overline{\text{Vect}}(\mathcal{O}_K)$ and $\text{pro}\overline{\text{Vect}}(\mathcal{O}_K)$ respectively.

As discussed in 5.3.2, we have natural isomorphisms which define the duality between the \mathcal{O}_K -linear categories $CTC_{\mathcal{O}_K}$ and $CP_{\mathcal{O}_K}$:

$$(6.16) \quad \text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\widehat{E}, F^\vee) \xrightarrow{\sim} \text{Hom}_{\mathcal{O}_K}(F, \widehat{E}^\vee).$$

Indeed both $\text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\widehat{E}, F^\vee)$ and $\text{Hom}_{\mathcal{O}_K}(F, \widehat{E}^\vee)$ may be identified with the \mathcal{O}_K -module consisting in the \mathcal{O}_K -bilinear maps

$$b : \widehat{E} \times F \longrightarrow \mathcal{O}_K$$

which are continuous in the first variable, or equivalently with the \mathcal{O}_K -module $\varinjlim_{U \in \mathcal{U}(\widehat{E})} E_U^\vee \otimes_{\mathcal{O}_K} F^\vee$.

Similarly, for any field embedding $\sigma : K \hookrightarrow \mathbb{C}$, we have a duality isomorphism of complex vector spaces:

$$(6.17) \quad \text{Hom}_{\mathbb{C}}^{\text{cont}}(\widehat{E}_\sigma, F_\sigma^\vee) \xrightarrow{\sim} \text{Hom}_{\mathbb{C}}(F_\sigma, \widehat{E}_\sigma^\vee).$$

Indeed, both $\text{Hom}_{\mathbb{C}}^{\text{cont}}(\widehat{E}_\sigma, F_\sigma^\vee)$ and $\text{Hom}_{\mathbb{C}}(F_\sigma, \widehat{E}_\sigma^\vee)$ may be identified with the complex vector space consisting in the \mathbb{C} -bilinear maps

$$\tilde{b} : \widehat{E}_\sigma \times F_\sigma \longrightarrow \mathbb{C}$$

which are continuous in the first variable, or equivalently with the complex vector space

$$\varinjlim_{U \in \mathcal{U}(\widehat{E})} E_{U,\sigma}^\vee \otimes_{\mathbb{C}} F_\sigma^\vee.$$

Moreover, for any $\lambda \in \mathbb{R}_+$, the bijection (6.17) defines by restriction a bijection:

$$(6.18) \quad \text{Hom}_{\mathbb{C}}^{\text{cont}}(\widehat{E}_{\sigma}, F_{\sigma}^{\vee}) \cap \text{Hom}_{\mathbb{C}}^{\leq \lambda}((E_{\sigma}^{\text{Hilb}}, \|\cdot\|_{E,\sigma}), (F_{\sigma}^{\vee}, \|\cdot\|_{F,\sigma}^{\vee})) \xrightarrow{\sim} \text{Hom}_{\mathbb{C}}^{\leq \lambda}((F_{\sigma}, \|\cdot\|_{F,\sigma}), (\widehat{E}_{\sigma}^{\vee}, \|\cdot\|_{E,\sigma}^{\vee})).$$

Indeed, the subsets of $\text{Hom}_{\mathbb{C}}^{\text{cont}}(\widehat{E}_{\sigma}, F_{\sigma}^{\vee})$ and of $\text{Hom}_{\mathbb{C}}(F_{\sigma}, \widehat{E}_{\sigma}^{\vee})$ defined by both sides of (6.18) may be identified with the spaces of \mathbb{C} -bilinear maps \tilde{b} as above such that, when considered as bilinear maps on $E_{\sigma}^{\text{Hilb}} \times F_{\sigma}$, their ε -norm

$$\|\tilde{b}\|_{\varepsilon} := \sup \left\{ |\tilde{b}(x, y)|; (x, y) \in E_{\sigma}^{\text{Hilb}} \times F_{\sigma}, \|x\|_{E,\sigma} \leq 1, \|y\|_{F,\sigma} \leq 1 \right\}$$

is at most λ .

Besides, the identifications (6.16) and (6.17) are compatible with “extension of scalars through $\sigma : K \hookrightarrow \mathbb{C}$.” In other words, (6.16) and (6.17) fits into a commutative diagram:

$$\begin{array}{ccc} \text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\widehat{E}, F^{\vee}) & \xrightarrow{\sim} & \text{Hom}_{\mathcal{O}_K}(F, \widehat{E}^{\vee}) \\ \downarrow \cdot \otimes_{\sigma} 1_{\mathbb{C}} & & \downarrow \cdot \otimes_{\sigma} 1_{\mathbb{C}} \\ \text{Hom}_{\mathbb{C}}^{\text{cont}}(\widehat{E}_{\sigma}, F_{\sigma}^{\vee}) & \xrightarrow{\sim} & \text{Hom}_{\mathbb{C}}(F_{\sigma}, \widehat{E}_{\sigma}^{\vee}). \end{array}$$

Together with the bijections (6.18), this shows that the bijection (6.16) defines — by restriction — a bijection

$$(6.19) \quad \text{Hom}_{\mathcal{O}_K}^{\leq \lambda}(\widehat{E}, \overline{F}^{\vee}) \xrightarrow{\sim} \text{Hom}_{\mathcal{O}_K}^{\leq \lambda}(\overline{F}, \widehat{E}^{\vee})$$

for every $\lambda \in \mathbb{R}_+$, and consequently an \mathcal{O}_K -linear isomorphism:

$$(6.20) \quad \text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\widehat{E}, \overline{F}^{\vee}) \xrightarrow{\sim} \text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\overline{F}, \widehat{E}^{\vee}).$$

We may finally formulate the duality between the categories of pro- and ind-hermitian vector bundles over $\text{Spec } \mathcal{O}_K$ as the following statement, to be compared with the duality between $CTC_{\mathcal{O}_K}$ and $CP_{\mathcal{O}_K}$ that constitutes the special case of Proposition 5.3.1 where $A = \mathcal{O}_K$:

Proposition 6.5.1. *The bijections (6.19) with $\lambda = 1$ and (6.20) define adjunctions of functors:*

$$\cdot^{\vee} : \text{pro}\overline{\text{Vect}}^{\leq 1}(\mathcal{O}_K) \rightleftarrows \text{ind}\overline{\text{Vect}}^{\leq 1}(\mathcal{O}_K)^{\text{op}} : \cdot^{\vee}$$

and

$$\cdot^{\vee} : \text{pro}\overline{\text{Vect}}(\mathcal{O}_K) \rightleftarrows \text{ind}\overline{\text{Vect}}(\mathcal{O}_K)^{\text{op}} : \cdot^{\vee}.$$

These are actually adjoint equivalences. Their unit and counit are natural isomorphisms η and ϵ defined by isometric isomorphisms

$$\epsilon_{\overline{F}} : \overline{F} \xrightarrow{\sim} \overline{F}^{\vee\vee} \quad \text{and} \quad \eta_{\widehat{E}} : \widehat{E} \xrightarrow{\sim} \widehat{E}^{\vee\vee},$$

associated to any object \overline{F} in $\text{ind}\overline{\text{Vect}}(\mathcal{O}_K)$ and any object \widehat{E} in $\text{pro}\overline{\text{Vect}}(\mathcal{O}_K)$, whose underlying morphisms from F to $F^{\vee\vee}$ and from \widehat{E} to $\widehat{E}^{\vee\vee}$ are the biduality isomorphisms:

$$\begin{array}{ccc} \epsilon_F : F & \xrightarrow{\sim} & \text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\text{Hom}_{\mathcal{O}_K}(F, \mathcal{O}_K), \mathcal{O}_K) \\ f & \mapsto & (\xi \mapsto \xi(f)) \end{array}$$

and

$$\begin{array}{ccc} \eta_{\widehat{E}} : \widehat{E} & \xrightarrow{\sim} & \text{Hom}_{\mathcal{O}_K}(\text{Hom}_{\mathcal{O}_K}^{\text{cont}}(\widehat{E}, \mathcal{O}_K), \mathcal{O}_K) \\ e & \mapsto & (\zeta \mapsto \zeta(e)). \end{array}$$

Proof. This may be deduced from the duality between $CTC_{\mathcal{O}_K}$ and $CP_{\mathcal{O}_K}$ established in Proposition 5.3.1, combined with basic results concerning the duality of (pre-)Hilbert spaces.

At this stage, we may also argue directly as follows. The naturality (in each of the relevant categories) with respect to \widehat{E} and \overline{F} of the bijections (6.19) with $\lambda = 1$ and (6.20) is straightforward.

We are left to show that the unit $\epsilon_{\overline{F}}$ and counit $\eta_{\widehat{E}}$ of these adjunctions are isometric isomorphisms with underlying isomorphisms the biduality isomorphisms ϵ_F and $\eta_{\widehat{E}}$.

This directly follows from (i) the validity of these properties when \overline{F} and \widehat{E} are hermitian vector bundles, (ii) the compatibility of the duality functors with inductive and projective limits (see (6.11) and (6.15)), and (iii) the fact that any ind- (resp. pro-)hermitian vector bundle over $\text{Spec } \mathcal{O}_K$ may be realized as an inductive (resp. projective) limit of an admissible system of hermitian vector bundles (*cf.* paragraphs 6.3.1 and 6.3.2). \square

6.6. Examples – I. Formal series and holomorphic functions on disks. Let R be a positive real number.

Let us consider the open disc in \mathbb{C} of radius R ,

$$D(R) := \{z \in \mathbb{C} \mid |z| < R\},$$

and the space $\mathcal{O}^{\text{an}}(D(R))$ of holomorphic functions on $D(R)$.

Equipped with the topology of uniform convergence on compact subsets of $D(R)$, it is a Fréchet space. Moreover, Taylor expansion at 0 defines an inclusion

$$\begin{aligned} i_R : \mathcal{O}^{\text{an}}(D(R)) &\hookrightarrow \mathbb{C}[[X]] \\ f &\longmapsto \sum_{n \in \mathbb{N}} (1/n!) f^{(n)}(0) X^n. \end{aligned}$$

When $\mathbb{C}[[X]] \simeq \mathbb{C}^{\mathbb{N}}$ is equipped with its natural Fréchet topology, defined by the simple convergence of coefficients, the map i_R is continuous with dense image.

Let f be an element of $\mathcal{O}^{\text{an}}(D(R))$ and let

$$f(z) = \sum_{n \in \mathbb{N}} a_n z^n$$

be its expansion at the origin.

For any $r \in [0, R[$, we have:

$$(6.21) \quad \int_0^1 |f(re^{2\pi it})|^2 dt = \sum_{n \in \mathbb{N}} r^{2n} |a_n|^2.$$

This relation shows that, for every $r \in]0, R[$, one defines a hermitian norm $\|\cdot\|_r$ on $\mathcal{O}^{\text{an}}(D(R))$ by letting:

$$\|f\|_r^2 = \int_0^1 |f(re^{2\pi it})|^2 dt,$$

and that the *Hardy space*

$$H^2(R) := \left\{ f \in \mathcal{O}^{\text{an}}(D(R)) \mid \sup_{r \in [0, R[} \|f\|_r^2 < +\infty \right\}$$

becomes a Hilbert space when equipped with the norm $\|\cdot\|_R$ defined by:

$$\|f\|_R^2 := \sup_{r \in [0, R[} \|f\|_r^2 = \sum_{n \in \mathbb{N}} R^{2n} |a_n|^2.$$

From (6.21), we also derive:

$$(6.22) \quad \|f\|_{L^2(D(R))} := \int_{x+iy \in D(R)} |f(x+iy)|^2 dx dy$$

$$(6.23) \quad = \int_0^R \int_0^{2\pi} |f(re^{i\theta})|^2 r dr d\theta$$

$$(6.24) \quad = \sum_{n \in \mathbb{N}} \pi(n+1)^{-1} R^{2(n+1)} |a_n|^2.$$

and, equipped with the norm $\|\cdot\|_{L^2(D(R))}$, the *Bergman space*

$$B(R) := \mathcal{O}^{\text{an}}(D(R)) \cap L^2(D(R))$$

is also a Hilbert space.

Observe that the expressions (6.21) and (6.22) for the norms $\|\cdot\|_R$ and $\|\cdot\|_{L^2(D(R))}$ show that the vector space $\mathbb{C}[T]$ is dense both in $H^2(R)$ and in $B(R)$. This show that the composite injections

$$i_R^H : H^2(R) \hookrightarrow \mathcal{O}^{\text{an}}(D(R)) \hookrightarrow \mathbb{C}[[X]]$$

and

$$i_R^B : B(R) \hookrightarrow \mathcal{O}^{\text{an}}(D(R)) \hookrightarrow \mathbb{C}[[X]],$$

that clearly are continuous, have dense images.

Moreover, $H^2(R)$ and in $B(R)$ are subspaces of $\mathcal{O}^{\text{an}}(D(R))$ invariant under the operation of complex conjugation $\bar{\cdot}$, defined by

$$\bar{f}(z) := \overline{f(\bar{z})} = \sum_{n \in \mathbb{N}} \overline{a_n} z^n.$$

Finally, we may define the following pro-hermitian vector bundles over $\text{Spec } \mathbb{Z}$:

$$(6.25) \quad \widehat{H}(R) := (\mathbb{Z}[[X]], (H^2(R), \|\cdot\|_R, i_R^H))$$

and

$$\widehat{B}(R) := (\mathbb{Z}[[X]], (B(R), \|\cdot\|_R, i_R^B)),$$

that may be seen as arithmetic avatars of the classical Hardy and Bergman spaces.

Let us emphasize that the isomorphism class in $\text{pro}\overline{\text{Vect}}(\mathcal{O}_K)$ (and *a fortiori* in $\text{pro}\overline{\text{Vect}}^{\leq 1}(\mathcal{O}_K)$) of $\widehat{H}(R)$, or of $\widehat{B}(R)$, varies with $R \in \mathbb{R}_+^*$. (This may be shown by considering their θ -invariants $h_\theta^i(\widehat{H}(R) \otimes \overline{\mathcal{O}}(\delta))$; see Proposition 7.4.3, *infra*.)

6.7. Examples. II – Injectivity and surjectivity of morphisms of pro-hermitian vector bundles. In this paragraph, we gather some observations and examples that demonstrate that, if

$$f : \widehat{E} \longrightarrow \widehat{F}$$

is a morphism of pro-hermitian vector bundles over $\text{Spec } \mathcal{O}_K$, the injectivity (resp. surjectivity) properties of the underlying morphisms $\hat{f} : \widehat{E} \longrightarrow \widehat{F}$, $\hat{f}_\sigma : \widehat{E}_\sigma \longrightarrow \widehat{F}_\sigma$ and $f_\sigma : E_\sigma^{\text{Hilb}} \longrightarrow F_\sigma^{\text{Hilb}}$ (of topological \mathcal{O}_K -modules and of complex Fréchet and Hilbert spaces) are in general loosely related.

6.7.1. Concerning their injectivity, on the positive side, let us observe that each of the following assertions implies the following one:

- (i) the morphism $\hat{f} : \hat{E} \longrightarrow \hat{F}$ of topological \mathcal{O}_K -module is injective and strict;
- (ii) the \mathbb{C} -linear map $\hat{f}_\sigma : \hat{E}_\sigma \longrightarrow \hat{F}_\sigma$ is injective;
- (iii) the \mathbb{C} -linear map $f_\sigma : E_\sigma^{\text{Hilb}} \longrightarrow F_\sigma^{\text{Hilb}}$ is injective.

Indeed, the implication (i) \implies (ii) follows from Proposition 5.4.12, and the implication (ii) \implies (iii) is clear.

Besides, the injectivity (ii) of \hat{f}_σ immediately implies the injectivity of \hat{f} .

6.7.2. Simple examples of morphisms of pro-hermitian vector bundles which demonstrate that \hat{f} or f_σ may be injective, while \hat{f}_σ is not, are easily obtained by the constructions in Proposition 5.5.1, 2), and Proposition 5.5.4.

Indeed, for any $a \in \mathbb{Z}$, the map ϕ_a considered in Proposition 5.5.4 with $A = \mathbb{Z}$ — namely the map from $\mathbb{Z}[[X]]$ to itself defined by the formula

$$(6.26) \quad \phi_a(f) := (1 - a/X)f + af(0)/X = f - a(f - f(0))/X$$

— defines a morphism between “arithmetic Hardy spaces” (as defined in (6.25))

$$\Phi_a : \widehat{H}(R) \longrightarrow \widehat{H}(R')$$

for any two positive real numbers R and R' such that $R' \leq R$. By definition, $\widehat{\Phi}_a := \phi_a$, and the morphisms

$$\phi_{a,\mathbb{C}} := \widehat{\Phi}_{a,\mathbb{C}} : \mathbb{C}[[X]] \longrightarrow \mathbb{C}[[X]]$$

and

$$\phi_a^{\text{Hilb}} := \Phi_{a,\mathbb{C}} : H^2(R) \longrightarrow H^2(R')$$

are still defined by formula (6.26), and actually make sense for any $a \in \mathbb{C}$.

According to Proposition 5.5.4, the map

$$\widehat{\Phi}_a := \phi_a : \mathbb{Z}[[X]] \longrightarrow \mathbb{Z}[[X]]$$

is injective if and only if $a \notin \{1, -1\}$, is not surjective, but satisfies $\phi_a(\mathbb{Z}[X]) = \mathbb{Z}[X]$. Moreover $\phi_{a,\mathbb{C}}$ satisfies $\phi_{a,\mathbb{C}}(\mathbb{C}[X]) = \mathbb{C}[X]$ and is injective if and only if $a = 0$.

These properties are complemented by the following proposition, that we leave as an easy exercise:

Proposition 6.7.1. *For any $a \in \mathbb{C}^*$, the map*

$$\phi_{a,\mathbb{C}} : \mathbb{C}[[X]] \longrightarrow \mathbb{C}[[X]]$$

is surjective and its kernel is the line $\mathbb{C} \cdot \sum_{k \in \mathbb{N}} a^{-k} X^k$.

For any $a \in \mathbb{C}$ and any two positive real numbers R and R' such that $R' \leq R$, the continuous linear map

$$\phi_a^{\text{Hilb}} : H^2(R) \longrightarrow H^2(R')$$

is injective if and only if $|a| \leq R$. When $|a| > R$, its kernel is the line $\mathbb{C} \cdot (a - X)^{-1}$. Moreover the following conditions are equivalent:

- (i) ϕ_a^{Hilb} is onto;
- (ii) ϕ_a^{Hilb} is a strict morphism of complex topological vector space;
- (iii) $R = R'$ and $|a| \neq R$.

□

In particular, we obtain:

Scholium 6.7.2. *For any $a \in \mathbb{Z} \setminus \{1, 0, -1\}$, and any $R \in \mathbb{R}_+^*$, the morphism of pro-hermitian vector bundles over $\text{Spec } \mathbb{Z}$*

$$\Phi_a : \widehat{H}(R) \longrightarrow \widehat{H}(R)$$

is such that $\widehat{\Phi}_a : \mathbb{Z}[[X]] \longrightarrow \mathbb{Z}[[X]]$ is injective and not strict, and $\widehat{\Phi}_{a\mathbb{C}} : \mathbb{C}[[X]] \longrightarrow \mathbb{C}[[X]]$ is strict and not injective. Moreover $\Phi_{a,\mathbb{C}} : H^2(R) \longrightarrow H^2(R)$ is an isomorphism (resp. injective with dense image, but not strict; resp. surjective but not injective) if $|a| < R$ (resp. if $|a| = R$; resp. if $|a| > R$). \square

6.7.3. Concerning the surjectivity properties of the underlying morphisms attached to a morphism of pro-hermitian vector bundles $\hat{f} : \widehat{E} \longrightarrow \widehat{F}$ over some arithmetic curve $\text{Spec } \mathcal{O}_K$, we may consider the following conditions, where we denote by σ some field embedding of K into \mathbb{C} :

- (i) the image $\hat{f}(\widehat{E})$ of the morphism $\hat{f} : \widehat{E} \longrightarrow \widehat{F}$ is dense in \widehat{F} ;
- (ii) the K -linear map $\hat{f}_K : \widehat{E}_K \longrightarrow \widehat{F}_\sigma$ is surjective;
- (iii) the \mathbb{C} -linear map $\hat{f}_\sigma : \widehat{E}_\sigma \longrightarrow \widehat{F}_\sigma$ is surjective;
- (iv) the image $f_\sigma(E_\sigma^{\text{Hilb}})$ of the continuous \mathbb{C} -linear map $f_\sigma : E_\sigma^{\text{Hilb}} \longrightarrow F_\sigma^{\text{Hilb}}$ is dense in F_σ^{Hilb} .

The properties of the morphisms in CTC_k when k is a field presented in paragraphs 5.3.3 and 5.4.4 (see notably Propositions 5.3.4 and 5.4.9) show that the morphisms \hat{f}_K and \hat{f}_σ in CTC_K and $CTC_\mathbb{C}$ are necessarily strict, and are therefore surjective if and only if they have a dense image. Proposition 5.3.4 also shows that the surjectivity of \hat{f}_K and of \hat{f}_σ are equivalent.

From these observations, one immediately derives the validity of the following implications:

$$(i) \implies (ii) \iff (iii) \iff (iv).$$

The converse implications $(ii) \implies (i)$ and $(iii) \implies (iv)$ do not hold in general. This will be demonstrated by the examples in 6.7.4 and 6.7.5 below.

6.7.4. Let R be a positive real number and let p be a prime number.

Let us consider the morphism

$$f : \widehat{H}(R) \longrightarrow \widehat{H}(R)$$

defined by the morphism of multiplication by $X - p$. Namely,

$$\hat{f} := (X - p) : \mathbb{Z}[[X]] \longrightarrow \mathbb{Z}[[X]]$$

is the morphism β_2 considered in Proposition 5.5.1, 2), and the continuous \mathbb{C} -linear map

$$f_\mathbb{C} := H^2(R) \longrightarrow H^2(R)$$

is the multiplication by the function $(z \mapsto z - p)$.

Then, according to Proposition 5.5.1, the cokernel of \hat{f} may be identified, as a topological \mathbb{Z} -module, with the p -adic integers \mathbb{Z}_p . Moreover, the maps $\hat{f}_\mathbb{Q} : \mathbb{Q}[[X]] \longrightarrow \mathbb{Q}[[X]]$ and $\hat{f}_\mathbb{C} : \mathbb{C}[[X]] \longrightarrow \mathbb{C}[[X]]$ are still defined by the multiplication by $X - p$, and are therefore isomorphisms.

Finally the injective map $f_\mathbb{C}$ is an isomorphism (resp. has a dense image, but is not surjective; resp. has a closed image of codimension 1) if $R < p$ (resp. if $R = p$; resp. if $R > p$).

6.7.5. We finally construct an example of a morphism $f : \widehat{E} \longrightarrow \widehat{F}$ of pro-hermitian vector bundles over $\text{Spec } \mathcal{O}_K$ such that \hat{f} (and consequently \hat{f}_K and \hat{f}_σ for every embedding $\sigma : K \hookrightarrow \mathbb{C}$) is an isomorphism, and f_σ is an isometry with infinite dimensional cokernel.

For simplicity, we shall assume that $K = \mathbb{Q}$.

Let $I = [a, b]$ be a closed bounded interval in \mathbb{R} , of length $b - a > 0$.

To I , we may attach the ind-hermitian vector bundle $\overline{V}_I := (V_I, \|\cdot\|_I)$ over $\text{Spec } \mathbb{Z}$ defined by the \mathbb{Z} -module $V_I := \mathbb{Z}[X]$ equipped with the L^2 -norm $\|\cdot\|_I$ on $V_{I,\mathbb{C}} = \mathbb{C}[X]$ defined by

$$\|P\|_I^2 := \int_I |P(t)|^2 dt.$$

The completion of the complex normed space $(V_I, \|\cdot\|_I)$ is the Hilbert space $L^2(I)$. Consequently, if we consider the pro-hermitian vector bundle

$$\widehat{F}_I := \overline{V}_I^\vee$$

dual to \overline{V}_I , its underlying Hilbert space $F_{I,\mathbb{C}}^{\text{Hilb}}$ may also be identified²⁷ with $L^2(I)$.

Let I' be another closed bounded interval of positive length, and let $\overline{V}_{I'}$ and

$$\widehat{F}_{I'} := \overline{V}_{I'}^\vee$$

be the associated ind- and pro-hermitian vector bundles over $\text{Spec } \mathbb{Z}$. Let us moreover assume that

$$I' \subset I.$$

The identity map $Id_{\mathbb{Z}[X]}$ defines a morphism $\rho_{I'I}$ in $\text{Hom}_{\mathbb{Z}}^{\leq 1}(\overline{V}_I, \overline{V}_{I'})$. Its adjoint map

$$\eta_{II'} := \rho_{I'I}^\vee \in \text{Hom}_{\mathbb{Z}}^{\leq 1}(\widehat{F}_{I'}, \widehat{F}_I)$$

is easily seen to be the morphism of pro-hermitian vector bundles over $\text{Spec } \mathbb{Z}$ defined by the morphism

$$\overline{\eta}_{II'} := Id_{\mathbb{Z}[X]}^\vee$$

of topological \mathbb{Z} -modules and the continuous \mathbb{C} -linear map “extension by zero”

$$\eta_{II',\mathbb{C}} : L^2(I') \longrightarrow L^2(I)$$

that sends a function $\Psi \in L^2(I')$ to the function $\eta_{II',\mathbb{C}}(\psi)$ such that

$$\begin{aligned} \eta_{II',\mathbb{C}}(\psi)(x) &:= \psi(x) & \text{if } x \in I' \\ &:= 0 & \text{if } x \in I \setminus I'. \end{aligned}$$

This map $\eta_{II',\mathbb{C}}$ is an isometry, and its cokernel may be identified with $L^2(I \setminus I')$, which is infinite dimensional if $I' \neq I$.

6.8. Examples. III – Subgroups of pre-Hilbert spaces and ind-euclidean lattices. Let $(H, \|\cdot\|)$ be a real pre-Hilbert space, and let Γ be a subgroup of $(H, +)$.

The inclusion morphism $i : \Gamma \longrightarrow H$ uniquely extends to a \mathbb{R} -linear map $i_{\mathbb{R}} : \Gamma_{\mathbb{R}} := \Gamma \otimes_{\mathbb{R}} \mathbb{R} \longrightarrow H$. Its image is

$$\text{im } i_{\mathbb{R}} = \sum_{f \in \Gamma} \mathbb{R} \cdot f.$$

Proposition 6.8.1. *The following two conditions are equivalent:*

- (i) *The map $i_{\mathbb{R}}$ is injective and the Γ is an object of $CP_{\mathbb{Z}}$.*
- (ii) *The group Γ is countable, and the following condition is satisfied:*
 - (F) *For any finite subset F of Γ the abelian group $\sum_{f \in F} \mathbb{Z} \cdot f$ has finite index in $\Gamma \cap \sum_{f \in F} \mathbb{R} \cdot f$.*

Recall that the objects of $CP_{\mathbb{Z}}$ are precisely the countable free \mathbb{Z} -modules.

Observe also that $i_{\mathbb{R}}$ is injective if and only if $\|i_{\mathbb{R}}(\cdot)\|$ is a prehilbertian norm on $\Gamma_{\mathbb{R}}$. Consequently, Condition (i) may be rephrased as:

- (i') *The pair $(\Gamma, \|i_{\mathbb{R}}(\cdot)\|)$ defines an ind-euclidean lattice.*

²⁷The natural isomorphism $L^2(I) \simeq F_{I,\mathbb{C}}^{\text{Hilb}}$ is characterized by the fact that the composite map $L^2(I) \simeq F_{I,\mathbb{C}}^{\text{Hilb}} \longrightarrow \widehat{F}_{I,\mathbb{C}} := \text{Hom}_{\mathbb{C}}(V_{I,\mathbb{C}}, \mathbb{C})$ sends a function $\phi \in L^2(I)$ to the linear form $(P \mapsto \int_I \phi(t)P(t) dt)$ on $V_{I,\mathbb{C}} = \mathbb{C}[X]$.

Proof. The direct implication (i) \Rightarrow (ii) is straightforward, since Condition **(F)** is satisfied by the subgroup $\Gamma = \mathbb{Z}^I$ of the real vector space $\Gamma_{\mathbb{R}} \simeq \mathbb{R}^I$, for any (countable) set I .

To establish the converse implication (ii) \Rightarrow (i), let us assume that (ii) is satisfied, and consider a sequence $(f_n)_{n \in \mathbb{N}} \in \Gamma^{\mathbb{N}}$ in which any element of Γ occurs.

For any $i \in \mathbb{N}$, let us consider

$$\Gamma_i := \Gamma \cap \sum_{0 \leq n \leq i} \mathbb{R}.f_i.$$

It is a torsion free \mathbb{Z} -module, which contains $\sum_{0 \leq n \leq i} \mathbb{Z}.f_i$ as a submodule of finite index. Consequently Γ_i is a finitely generated torsion free \mathbb{Z} -module. Moreover, for any $i \in \mathbb{N}$, Γ_i is clearly a saturated \mathbb{Z} -submodule of Γ_{i+1} , and $\Gamma = \bigcup_{i \in \mathbb{N}} \Gamma_i$.

Therefore, the \mathbb{Z} -module Γ is countably generated and projective, hence isomorphic to $\mathbb{Z}^{(I)}$, with I at most countable (this follows from the implications (4) \Rightarrow (1) and (3) in Proposition 5.1.1). We may assume that

$$I = \{n \in \mathbb{N} \mid n < N\}$$

with $N := |I| \in \mathbb{N} \cup \{+\infty\}$.

Let $(e_i)_{i \in I}$ be a \mathbb{Z} -basis of Γ . It is also a \mathbb{R} -basis of $\Gamma_{\mathbb{R}}$, and to prove the injectivity of $i_{\mathbb{R}}$, we are left to show that the family $(e_i)_{i \in I}$, considered as a family of vectors in H , is free over \mathbb{R} .

To achieve this, observe that, for every $n \in I$, the \mathbb{Z} -module $\sum_{0 \leq i < n} \mathbb{Z}.e_i$ is a saturated submodule of Γ , hence a saturated submodule of $\Gamma \cap \sum_{0 \leq i < n} \mathbb{R}.e_i$. Besides, the validity of **(F)** implies that $\sum_{0 \leq i < n} \mathbb{Z}.e_i$ has finite index in $\Gamma \cap \sum_{0 \leq i < n} \mathbb{R}.e_i$. Consequently, these two \mathbb{Z} -modules coincide:

$$(6.27) \quad \sum_{0 \leq i < n} \mathbb{Z}.e_i = \Gamma \cap \sum_{0 \leq i < n} \mathbb{R}.e_i.$$

If the family $(e_i)_{i \in I}$ were not free over \mathbb{R} , there would exist $n \in I$ such that $e_n \in \sum_{0 \leq i < n} \mathbb{R}.e_i$. This would contradict (6.27). \square

Proposition 6.8.2. *If Γ is a discrete subgroup in the pre-Hilbert space $(H, \|\cdot\|)$, then, for any finite subset F of Γ , the subgroups $\sum_{f \in F} \mathbb{Z}.f$ and $\Gamma \cap \sum_{f \in F} \mathbb{R}.f$ are lattices in the finite dimensional real vector space $\sum_{f \in F} \mathbb{R}.f$.*

Proof. The subgroups $\sum_{f \in F} \mathbb{Z}.f$ and $\Gamma \cap \sum_{f \in F} \mathbb{R}.f$ satisfy the inclusion

$$\sum_{f \in F} \mathbb{Z}.f \subset \Gamma \cap \sum_{f \in F} \mathbb{R}.f.$$

Besides $\sum_{f \in F} \mathbb{Z}.f$ generates the real vector space $\sum_{f \in F} \mathbb{R}.f$, and $\Gamma \cap \sum_{f \in F} \mathbb{R}.f$ is discrete in this vector space, since Γ is discrete in H . Therefore both $\sum_{f \in F} \mathbb{Z}.f$ and $\Gamma \cap \sum_{f \in F} \mathbb{R}.f$ generate and are discrete in $\sum_{f \in F} \mathbb{R}.f$. \square

From Propositions 6.8 and 6.8.2, we immediately derive:

Corollary 6.8.3. *If the pre-Hilbert space $(H, \|\cdot\|)$ is separable and if Γ is discrete in H , then the pair $(\Gamma, \|i_{\mathbb{R}}(\cdot)\|)$ defines an ind-euclidean lattice.* \square

7. θ -INVARIANTS OF INFINITE DIMENSIONAL HERMITIAN VECTOR BUNDLES: DEFINITIONS AND FIRST PROPERTIES

In this section, we extend the definitions and some of the basic properties h_{θ}^0 and h_{θ}^1 to infinite-dimensional vector bundles over an arithmetic curve $\text{Spec } \mathcal{O}_K$. As in the previous section, many constructions will be of a rather formal nature, and details could be skipped at first reading.

We should however emphasize that the extension of h_θ^0 to general pro-hermitian vector bundles cannot be completely formal. Indeed, as shown by Proposition 7.2.1 and the examples in paragraph 7.4.1, if some pro-hermitian vector \widehat{E} is realized as the projective limit $\varprojlim_i \overline{E}_i$ of some projective system

$$\overline{E}_\bullet : \overline{E}_0 \xleftarrow{q_0} \overline{E}_1 \xleftarrow{q_1} \dots \xleftarrow{q_{i-1}} \overline{E}_i \xleftarrow{q_i} \overline{E}_{i+1} \xleftarrow{q_{i+1}} \dots$$

of surjective admissible morphisms of hermitian vector bundles over $\text{Spec } \mathcal{O}_K$, then the sequence $(h_\theta^0(\overline{E}_i))_{i \in \mathbb{N}}$ is in general not convergent in $[0, +\infty]$; moreover, when it converges, its limit may depend of the chosen projective system \overline{E}_\bullet . This issue motivates our introduction of *two* generalizations, denoted by \underline{h}_θ^0 and \overline{h}_θ^0 , of h_θ^0 to the infinite dimensional setting.

The last paragraphs of this section are again dedicated to concrete examples involving these invariants. Notably we pursue the study of the “arithmetic Hardy spaces” $\widehat{H}(R)$ introduced in the previous section.

We still denote by K a number field, by \mathcal{O}_K its ring of integers, and by $\pi : \text{Spec } \mathcal{O}_K \rightarrow \text{Spec } \mathbb{Z}$ the morphism of schemes from $\text{Spec } \mathcal{O}_K$ to $\text{Spec } \mathbb{Z}$.

7.1. Limits of θ -invariants. It is straightforward that, for any ind-hermitian vector bundle \overline{F} over $\text{Spec } \mathcal{O}_K$, the following limit exists in $[0, +\infty]$:

$$(7.1) \quad h_\theta^0(\overline{F}) := \lim_{F' \in \mathcal{F}(F)} h_\theta^0(\overline{F}').$$

Indeed $h_\theta^0(\overline{F}')$ is an increasing function $F' \in \mathcal{F}(F)$ and this limit is actually given by the expression:

$$h_\theta^0(\overline{F}) := \log \sum_{v \in F} e^{-\pi \|v\|_{\pi_* \overline{F}}^2}$$

(where, by convention $\log(+\infty) = +\infty$).

Observe also the equality:

$$(7.2) \quad h_\theta^0(\overline{F}) := \sup_{F' \in \mathcal{F}(F)} h_\theta^0(\overline{F}').$$

In particular, if

$$\overline{F}_0 \hookrightarrow \overline{F}_1 \hookrightarrow \dots \hookrightarrow \overline{F}_i \hookrightarrow \overline{F}_{i+1} \hookrightarrow \dots$$

is an inductive system of admissible injections of hermitian vector bundles,

$$(7.3) \quad h_\theta^0(\varinjlim_i \overline{F}_i) = \lim_{i \rightarrow +\infty} h_\theta^0(\overline{F}_i).$$

Dually, for any pro-hermitian vector bundle $\widehat{E} := (\widehat{E}, (\overline{E}_U)_{U \in \mathcal{U}(\widehat{E})})$ over $\text{Spec } \mathcal{O}_K$, $h_\theta^1(\overline{E}_U)$ is a decreasing function of $U \in \mathcal{U}(\widehat{E})$, and we define:

$$(7.4) \quad h_\theta^1(\tilde{E}) := \lim_{U \in \mathcal{U}(\widehat{E})} h_\theta^1(\overline{E}_U).$$

It is again an element of $[0, +\infty]$, which, according to its very definition satisfies the following duality relation:

Proposition 7.1.1. *Let \overline{F} be an ind-hermitian vector bundle over $\text{Spec } \mathcal{O}_K$, and let $\widehat{E} := \overline{F}^\vee$ denote the dual pro-hermitian vector bundle. Then the following equality holds in $[0, +\infty]$.*

$$h_\theta^1(\widehat{E}) = h_\theta^0(\overline{F}).$$

□

By duality, the relation (7.2) becomes:

$$(7.5) \quad h_\theta^1(\widehat{E}) := \sup_{U \in \mathcal{U}(\widehat{E})} h_\theta^1(\overline{E}_U),$$

and (7.3) shows that, for any projective system

$$\overline{E}_0 \longleftarrow \overline{E}_1 \longleftarrow \dots \longleftarrow \overline{E}_i \longleftarrow \overline{E}_{i+1} \longleftarrow \dots$$

of admissible surjections of hermitian vector bundles, we have

$$(7.6) \quad h_\theta^1(\varprojlim_i \overline{E}_i) = \lim_{i \rightarrow +\infty} h_\theta^1(\overline{E}_i).$$

The proof of the following proposition is left as an easy exercise:

Proposition 7.1.2. *Let $\overline{F} := (F, \|\cdot\|)$ be an ind-hermitian vector bundle over $\text{Spec } \mathbb{Z}$, and let*

$$N_{\overline{F}} : \mathbb{R}_+ \longrightarrow \mathbb{N}_{>0} \cup \{+\infty\}$$

be the non-decreasing function defined by

$$N_{\overline{F}}(x) := |\{f \in F \mid \|f\| \leq x\}|.$$

If $h_\theta^0(\overline{F} \otimes \overline{\mathcal{O}}(\lambda_0))$ is finite for some $\lambda_0 \in \mathbb{R}$, then $h_\theta^0(\overline{F} \otimes \overline{\mathcal{O}}(\lambda))$ is finite for every $\lambda \in]-\infty, \lambda_0]$. Moreover, the following three conditions are equivalent:

- (i) *For any $\lambda \in \mathbb{R}$, $h_\theta^0(\overline{F} \otimes \overline{\mathcal{O}}(\lambda))$ is finite.*
- (ii) *For every $t \in \mathbb{R}_+^*$,*

$$\sum_{f \in F} e^{-\pi t \|f\|^2} < +\infty.$$

- (iii) *For any $x \in \mathbb{R}_+$, $N_{\overline{F}}(x)$ is finite and, when x goes to $+\infty$,*

$$\log N_{\overline{F}}(x) = o(x^{1/2}).$$

□

7.2. Upper and lower θ -invariants. In the applications to Diophantine geometry developed in the sequel to this article, the significant extension of θ -invariants to infinite rank hermitian vector bundles is not the “obvious” ones, simply defined as limits, that we introduced in the previous paragraph 7.1. Instead, a key role will be played by some avatar of the invariant h_θ^0 attached to a *pro*-hermitian vector bundle (or dually, by some avatar of h_θ^1 attached to an *ind*-hermitian vector bundle).

7.2.1. Upper limits of θ -invariants. As demonstrated by the next proposition, if we want them to take finite values on some infinite rank hermitian vector bundles, these generalized invariants cannot be defined by the obvious modifications of the definitions (7.4) and (7.1), where h_θ^1 would be replaced by h_θ^0 and *vice versa*.

Proposition 7.2.1. *For any pro-hermitian vector bundle $\widehat{E} := (\widehat{E}, (\overline{E}_U)_{U \in \mathcal{U}(\widehat{E})})$ of infinite rank over $\text{Spec } \mathcal{O}_K$, we have:*

$$(7.7) \quad \limsup_{U \in \mathcal{U}(\widehat{E})} h_\theta^0(\overline{E}_U) = +\infty.$$

For any ind-hermitian vector bundle \overline{F} of infinite rank over $\text{Spec } \mathcal{O}_K$, we have:

$$(7.8) \quad \limsup_{F' \in \mathcal{FS}(\overline{F})} h_\theta^1(\overline{F}') = +\infty.$$

Proof. The proof will be based on the following classical fact:

Lemma 7.2.2. *Let \hat{V} be a hermitian vector bundle of rank at least two over $\text{Spec } \mathcal{O}_K$.*

The saturated \mathcal{O}_K -submodules W of rank $\text{rk } V - 1$ in V are in bijection with the K -points of $\mathbb{P}(V_K)$ by the map which associates the quotient map $V_K \rightarrow V_K/W_K$ to W .

Moreover one defines a height function $h_{\mathbb{P}(V_K)} : \mathbb{P}(V_K)(K) \rightarrow \mathbb{R}$ associated to the line bundle $\mathcal{O}(1)$ over the projective space $\mathbb{P}(V_K)$ by the formula:

$$h_{\mathbb{P}(V_K)}(W) := \widehat{\deg} \overline{V/W}.$$

In particular, there exists saturated \mathcal{O}_K -submodules W of rank $\text{rk } V - 1$ in V such that $\widehat{\deg} \overline{V/W}$ take arbitrary large positive values. \square

Consider a pro-hermitian vector bundle \widehat{E} of infinite rank over $\text{Spec } \mathcal{O}_K$ and U a element of $\mathcal{U}(\widehat{E})$. We want to prove that there exists $U' \in \mathcal{U}(\widehat{E})$ contained in U with $h_\theta^0(\overline{E}_{U'})$ arbitrary large.

As \widehat{E} has infinite rank, there exists \tilde{U} in $\mathcal{U}(\widehat{E})$ such that $p_{\tilde{U}} : E_{\tilde{U}} \rightarrow E_U$ has a kernel $V := \ker p_{\tilde{U}}$ of rank at least 2. Let W be a saturated submodule of rank $\text{rk } V - 1$ in V . The inverse image

$$U' := q_{\tilde{U}}^{-1}(W)$$

of the saturated submodule W of $E_{\tilde{U}}$ by the quotient map $q_{\tilde{U}} : \widehat{E} \rightarrow \widehat{E}_{\tilde{U}}$ is an element of $\mathcal{U}(\widehat{E})$ contained in U . Moreover $\overline{E}_{U'}$ may be identified with the quotient $\overline{E}_U/\overline{W}$, which itself contains $\overline{V}/\overline{W}$. Therefore:

$$h_\theta^0(\overline{E}_{U'}) = h_\theta^0(\overline{E}_U/\overline{W}) \geq h_\theta^0(\overline{V}/\overline{W}) \geq \widehat{\deg} \overline{V/W}.$$

According to Lemma 7.2.2, this Arakelov degree takes an arbitrary large positive value for a suitable choice of W .

This completes the proof of (7.7). The formula (7.8) follows, by duality, from (7.7) applied to $\widehat{E} := \overline{F}^\vee$. \square

7.2.2. Definitions. In spite of the “divergent behavior” of the θ -invariants highlighted in Proposition 7.2.1, it is possible to introduce some “lower versions” \underline{h}_θ^i and some “upper versions” \overline{h}_θ^i of these invariants, that will belong to $[0, +\infty]$ in general, but will achieve finite values in significant instances.

For any pro-hermitian vector bundle $\widehat{E} := (\widehat{E}, (\overline{E}_U)_{U \in \mathcal{U}(\widehat{E})})$ over $\text{Spec } \mathcal{O}_K$, we may consider the class $\mathcal{L}(\widehat{E})$ of pairs (\overline{E}', ι) , where \overline{E}' is some hermitian vector bundle over $\text{Spec } \mathcal{O}_K$ and ι an element of $\text{Hom}_{\mathcal{O}_K}^{\leq 1}(\overline{E}', \widehat{E})$ such that $\iota : E' \rightarrow \widehat{E}$ is injective.

We may also consider the sub-class $\mathcal{L}_{\text{ad}}(\widehat{E})$ of $\mathcal{L}(\widehat{E})$ of the pairs (\overline{E}', ι) as above such that moreover the image $\iota(E')$ is a saturated \mathcal{O}_K -submodule of \widehat{E} and, for every embedding $\sigma : K \hookrightarrow \mathbb{C}$, the maps ι_σ is an isometry from $(E'_\sigma, \|\cdot\|_{\overline{E}', \sigma})$ to E_σ^{Hilb} equipped with its natural Hilbert norm $\|\cdot\|_{E_\sigma^{\text{Hilb}}}$, defined by (6.1).

Then we may introduce the following definitions:

$$(7.9) \quad \underline{h}_\theta^0(\widehat{E}) := \sup\{h_\theta^0(\overline{E}'); (\overline{E}', \iota) \in \mathcal{L}(\widehat{E})\}$$

and

$$(7.10) \quad \overline{h}_\theta^0(\widehat{E}) := \liminf_{U \in \mathcal{U}(\widehat{E})} h_\theta^0(\overline{E}_U).$$

Let \overline{F} denote some ind-hermitian vector bundle over $\text{Spec } \mathcal{O}_K$. We recall that we denote by $\text{coFS}(F)$ the family of \mathcal{O}_K -submodules F' of F such that F/F' is finitely generated and torsion free, and by $\mathcal{FS}(F)$ (resp. by $\mathcal{FS}(F)$) the family of finitely generated (resp. finitely generated and saturated) \mathcal{O}_K -submodules of F (cf. Section 5.1.1 *supra*).

We may also consider the following subset of $\text{co}\mathcal{FS}(F)$:

$$\mathcal{FQ}(\overline{F}) := \left\{ F' \in \text{co}\mathcal{FS}(F) \mid \text{for every } \sigma : K \hookrightarrow \mathbb{C}, F'_\sigma \text{ is closed in the pre-hilbert space } (F_\sigma, \|\cdot\|_{\overline{F}, \sigma}) \right\}.$$

For any $F \in \mathcal{FQ}(\overline{F})$, the finitely generated projective \mathcal{O}_K -module F/F' becomes a hermitian vector bundle over $\text{Spec } \mathcal{O}_K$, that we shall denote by $\overline{F/F'}$, when equipped with the quotient norms of the norms $\|\cdot\|_{\overline{F}, \sigma}$.

Using this notation, we may also define:

$$(7.11) \quad \underline{h}_\theta^1(\overline{F}) := \sup \{ h_\theta^1(\overline{F/F'}); F' \in \mathcal{FQ}(\overline{F}) \}$$

and

$$(7.12) \quad \overline{h}_\theta^1(\overline{F}) := \liminf_{F' \in \mathcal{F}(F)} h_\theta^1(\overline{F'}).$$

7.2.3. Variants. The following two propositions provide alternative definitions of the invariants $\underline{h}_\theta^0, \overline{h}_\theta^0, \underline{h}_\theta^1$, and \overline{h}_θ^1 that we have just introduced.

Proposition 7.2.3. *For any pro-hermitian vector bundle \widehat{E} over $\text{Spec } \mathcal{O}_K$, we have:*

$$(7.13) \quad \begin{aligned} \underline{h}_\theta^0(\widehat{E}) &= \sup \left\{ h_\theta^0(\overline{E'}); (\overline{E'}, \iota) \in \mathcal{L}_{\text{ad}}(\widehat{E}) \right\} \\ &= \sup \left\{ h_\theta^0(\overline{P}); P \text{ finitely generated } \mathcal{O}_K\text{-submodule of } \widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}} \right\} \\ &= \sup \left\{ h_\theta^0(\overline{P}); P \text{ finitely generated saturated } \mathcal{O}_K\text{-submodule of } \widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}} \right\}. \end{aligned}$$

Consequently, we also have:

$$(7.14) \quad \underline{h}_\theta^0(\widehat{E}) = \underline{h}_\theta^0(\pi_* \widehat{E}) = \log \sum_{v \in \widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}} e^{-\pi \|v\|_{\pi_* \widehat{E}}^2}.$$

In the second and third lines of (7.13), we have denoted by \overline{P} the hermitian vector bundle over $\text{Spec } \mathcal{O}_K$ defined by the finitely generated projective \mathcal{O}_K -module P equipped with the restrictions of the Hilbert norms $\|\cdot\|_{E_\sigma^{\text{Hilb}}}$ on the Hilbert spaces E_σ^{Hilb} . Observe also that $\widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}$ is a saturated \mathcal{O}_K -submodule of \widehat{E} , and that consequently, a submodule of $\widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}$ is saturated in $\widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}$ if and only if it is saturated in \widehat{E} .

Proof. The relations (7.13) directly follows from the definition of $\underline{h}_\theta^0(\widehat{E})$, from the increasing character of the θ -invariant h_θ^0 for hermitian vector bundles (see Proposition 3.3.2, 1)), and from the properties of the saturation of finitely generated \mathcal{O}_K -submodules of objects in $CTC_{\mathcal{O}_K}$ (see Corollary 5.4.2). We leave the details to the reader.

The equality

$$(7.15) \quad \underline{h}_\theta^0(\widehat{E}) = \log \sum_{v \in \widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}} e^{-\pi \|v\|_{\pi_* \widehat{E}}^2},$$

then follows from the second line in (7.13). Indeed, by the very definition of h_θ^0 , we have:

$$h_\theta^0(\overline{P}) = \log \sum_{v \in P} e^{-\pi \|v\|_{\pi_* \widehat{E}}^2},$$

and any finite subset of $\widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}$ is contained in some finitely generated \mathcal{O}_K -submodule of $\widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}$.

Applied to the pro-hermitian vector bundle $\pi_* \widehat{E}$ over $\text{Spec } \mathbb{Z}$, the equality (7.15) becomes

$$\underline{h}_\theta^0(\pi_* \widehat{E}) = \log \sum_{v \in \widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}} e^{-\pi \|v\|_{\pi_* \widehat{E}}^2}.$$

This completes the proof of (7.14). \square

Observe that, with the notation of Proposition 7.2.3, we also have, as a straightforward consequence of the definition of \overline{h}_θ^0 :

$$(7.16) \quad \overline{h}_\theta^0(\widehat{E}) \geq \overline{h}_\theta^0(\pi_* \widehat{E}).$$

Proposition 7.2.4. *For any ind-hermitian vector bundle \overline{F} over $\text{Spec } \mathcal{O}_K$, we have:*

$$(7.17) \quad \overline{h}_\theta^1(\overline{F}) = \liminf_{F' \in \mathcal{FS}(F)} h_\theta^1(F').$$

Proof. The inequality

$$\overline{h}_\theta^1(\overline{F}) \leq \liminf_{F' \in \mathcal{FS}(F)} h_\theta^1(F')$$

is clear.

To prove the converse inequality, simply observe that, for any submodule $F' \in \mathcal{FS}(F)$, its saturation F'^{sat} belongs to $\mathcal{FS}(F)$ and satisfies $h_\theta^1(\overline{F'^{\text{sat}}}) \leq h_\theta^1(F')$. \square

The expression (7.14) shows that the finiteness of $\underline{h}_\theta^0(\widehat{E})$ implies the countability of $\widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}$. The second half of the following easy proposition is a refinement of this observation:

Proposition 7.2.5. *For any ind-hermitian vector bundle \overline{F} over $\text{Spec } \mathcal{O}_K$ and any $\lambda \in \mathbb{R}_+$, we have:*

$$(7.18) \quad |\{x \in F \mid \|x\|_{\pi_* \overline{F}} \leq \lambda\}| \leq \exp(h_\theta^0(\overline{F}) + \pi \lambda^2).$$

For any pro-hermitian vector bundle \widehat{E} over $\text{Spec } \mathcal{O}_K$ and for any $\lambda \in \mathbb{R}_+$, we have:

$$(7.19) \quad |\{x \in \widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}} \mid \|x\|_{\pi_* \widehat{E}} \leq \lambda\}| \leq \exp(\underline{h}_\theta^0(\widehat{E}) + \pi \lambda^2).$$

Proof. Let us consider a finite subset A of $\{x \in F \mid \|x\|_{\pi_* \overline{F}} \leq \lambda\}$. The following inequalities are straightforward:

$$|A| \cdot \exp(-\pi \lambda^2) \leq \sum_{v \in A} e^{-\pi \|v\|_{\pi_* \overline{F}}^2} \leq \sum_{v \in F} e^{-\pi \|v\|_{\pi_* \overline{F}}^2} =: \exp(h_\theta^0(\overline{F})).$$

Their validity for an arbitrary A as above establishes (7.18).

The proof of (7.19) is similar to the one of (7.18). \square

7.3. Basic properties.

7.3.1. Duality. The newly defined θ -invariants of ind- and pro-hermitian vector bundles still satisfy a duality relation:

Proposition 7.3.1. *Let \overline{F} be an ind-hermitian vector bundle over $\text{Spec } \mathcal{O}_K$, and let $\widehat{E} := \overline{F}^\vee$ denote the dual pro-hermitian vector bundle. Then the following equalities hold in $[0, +\infty]$:*

$$(7.20) \quad \underline{h}_\theta^1(\overline{F}) = \underline{h}_\theta^0(\widehat{E}) \quad \text{and} \quad \overline{h}_\theta^1(\overline{F}) = \overline{h}_\theta^0(\widehat{E}).$$

Proof. According to the relations (7.11) and (7.13), the first equality may be written:

$$(7.21) \quad \sup \left\{ h_\theta^1(\overline{F/F'}); F' \in \mathcal{FQ}(\overline{F}) \right\} = \sup \left\{ h_\theta^0(\overline{P}); P \text{ finitely generated saturated } \mathcal{O}_K\text{-submodule of } \widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}} \right\}.$$

and, according to (7.17) and (7.10), the second one may be written:

$$(7.22) \quad \liminf_{F' \in \mathcal{FS}(F)} h_\theta^1(\overline{F'}) = \liminf_{U \in \mathcal{U}(\widehat{E})} h_\theta^0(\overline{E}_U).$$

To establish (7.22), recall that, according to Corollary 5.4.8, there is an inclusion reversing bijection

$$.\perp : \mathcal{FS}(F) \xrightarrow{\sim} \mathcal{U}(\widehat{E}).$$

and for any $U \in \mathcal{U}(\widehat{E})$, there is a canonical isomorphism of finitely generated projective \mathcal{O}_K -modules:

$$(7.23) \quad I_U : E_U = \widehat{E}/U \xrightarrow{\sim} (U^\perp)^\vee$$

(see diagram (5.29)). This isomorphism is easily seen to define an isometric isomorphism between the hermitian vector bundles \overline{E}_U and $\overline{U^\perp}^\vee$. Consequently:

$$h_\theta^0(\overline{E}_U) = h_\theta^0(\overline{U^\perp}^\vee) = h_\theta^1(\overline{U^\perp}).$$

The equality (7.22) follows by taking the lower limit over U in the filtering ordered set $\mathcal{U}(\widehat{E})$.

The proof of (7.21) is similar, and we shall leave some details to the reader. We may define an inclusion reversing bijection

$$.\perp : \mathcal{FQ}(\overline{F}) \xrightarrow{\sim} \left\{ \text{finitely generated saturated } \mathcal{O}_K\text{-submodule of } \widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}} \right\}$$

by sending a submodule F' in $\mathcal{FQ}(\overline{F})$ to

$$F'^\perp := \left\{ \xi \in \widehat{E} := F^\vee \mid \xi|_{F'} = 0 \right\}.$$

Moreover, for any $F' \in \mathcal{FQ}(\overline{F})$, the \mathcal{O}_K -linear map

$$\begin{array}{ccc} F'^\perp & \longrightarrow & (F/F')^\vee \\ \xi & \longmapsto & ([x] \mapsto \xi(x)) \end{array}$$

defines an isomorphism of hermitian vector bundles over $\text{Spec } \mathcal{O}_K$:

$$\overline{F'} \xrightarrow{\sim} \overline{F/F'}^\vee.$$

In particular, we have:

$$h_\theta^1(\overline{F/F'}) = h_\theta^0(\overline{F/F'}^\vee) = h_\theta^0(\overline{F'^\perp}).$$

The equality (7.21) follows by taking the supremum over F' in $\mathcal{FQ}(\overline{F})$. □

Corollary 7.3.2. *For any ind-hermitian vector bundle \overline{F} over $\text{Spec } \mathcal{O}_K$,*

$$(7.24) \quad \underline{h}_\theta^1(\overline{F}) = \underline{h}_\theta^1(\pi_* \overline{F}) \quad \text{and} \quad \overline{h}_\theta^1(\overline{F}) \geq \overline{h}_\theta^1(\pi_* \overline{F}).$$

Proof. This follows from the duality relations (7.20) and from the similar properties (7.14) and (7.16) satisfied by \underline{h}_θ^0 and \overline{h}_θ^0 . □

7.3.2. *Additivity. Comparing \underline{h}_θ^0 and \overline{h}_θ^0 .* Observe that the expression (7.14) for $\underline{h}_\theta^0(\widehat{E})$ implies that \underline{h}_θ^0 is an additive invariant of pro-hermitian vector bundles, and dually that \underline{h}_θ^1 is an additive invariant of ind-hermitian vector bundles:

Corollary 7.3.3. *For any finite family $(\widehat{E}_i)_{1 \leq i \leq n}$ (resp. $(\overline{F}_i)_{1 \leq i \leq n}$) of pro-hermitian (resp. of ind-hermitian) vector bundles over $\text{Spec } \mathcal{O}_K$,*

$$(7.25) \quad \underline{h}_\theta^0\left(\bigoplus_{1 \leq i \leq n} \widehat{E}_i\right) = \sum_{1 \leq i \leq n} \underline{h}_\theta^0(\widehat{E}_i) \quad (\text{resp. } \underline{h}_\theta^1\left(\bigoplus_{1 \leq i \leq n} \overline{F}_i\right) = \sum_{1 \leq i \leq n} \underline{h}_\theta^1(\overline{F}_i)).$$

□

Proposition 7.3.4. *For any pro-hermitian vector bundle \widehat{E} over $\text{Spec } \mathcal{O}_K$,*

$$(7.26) \quad \underline{h}_\theta^0(\widehat{E}) \leq \overline{h}_\theta^0(\widehat{E}).$$

For any ind-hermitian vector bundle \overline{F} over $\text{Spec } \mathcal{O}_K$,

$$(7.27) \quad \underline{h}_\theta^1(\overline{F}) \leq \overline{h}_\theta^1(\overline{F}).$$

Proof. To prove (7.26), let us consider a finitely generated \mathcal{O}_K -submodule P of $\widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}$.

For U small enough in $\mathcal{U}(\widehat{E})$, the quotient morphism $p_U : \widehat{E} \rightarrow E_U$ has an injective restriction:

$$p_U|_P : P \hookrightarrow E_U$$

(see Proposition 5.4.1). Moreover, for any x in P and for any embedding $\sigma : K \hookrightarrow \mathbb{C}$, the norm $\|p_U(x)\|_{\overline{E}_{U,\sigma}}$ is a non-decreasing function of $U \in \mathcal{U}(\widehat{E})$ and converges to $\|x\|_{E_{\sigma}^{\text{Hilb}}}$ when U becomes arbitrarily small.

This shows that

$$h_\theta^0(\overline{P}) = \lim_{U \in \mathcal{U}(\widehat{E})} h_\theta^0(\overline{p_U(P)}).$$

Together with the obvious inequality

$$h_\theta^0(\overline{p_U(P)}) \leq h_\theta^0(\overline{E}_U),$$

this implies the estimate

$$h_\theta^0(\overline{P}) \leq \liminf_{U \in \mathcal{U}(\widehat{E})} h_\theta^0(\overline{E}_U)$$

and establishes (7.26).

The second inequality (7.27) follows from (7.26) and from the duality relations (7.20). □

The possible existence of pro-hermitian vector bundles \overline{E} such that

$$(7.28) \quad \underline{h}_\theta^0(\widehat{E}) < \overline{h}_\theta^0(\widehat{E})$$

is an intriguing issue. A related issue is the additivity of \overline{h}_θ^0 , namely the validity of the first part of (7.25) for \overline{h}_θ^0 instead of \underline{h}_θ^0 .²⁸

In the geometric situation where one deals with pro-vector bundles \widehat{E} over a smooth projective curve C over some field k , pro-vector bundles \widehat{E} whose invariants $h^0(C, \widehat{E})$ and $\overline{h}^0(C, \widehat{E})$ — which

²⁸With the notation of Corollary 7.3.3, the inequality $\overline{h}_\theta^0(\bigoplus_{1 \leq i \leq n} \widehat{E}_i) \leq \sum_{1 \leq i \leq n} \overline{h}_\theta^0(\widehat{E}_i)$ holds as a straightforward consequence of the definition of \overline{h}_θ^0 and of the additivity of \underline{h}_θ^0 stated in Proposition 3.5.1. The additivity of \underline{h}_θ^0 shows that examples of pro-hermitian vector bundles \widehat{E}_i for which this inequality is strict would lead to examples where (7.28) is strict.

are the geometric counterparts of the invariants $\underline{h}_\theta^0(\overline{E})$ and $\overline{h}_\theta^0(\overline{E})$ of a pro-hermitian vector bundle — satisfy

$$h^0(C, \widehat{E}) < \overline{h}^0(C, \widehat{E})$$

may be constructed when C is an elliptic curve and, at least under some mild technical assumption, on any C of genus $g > 1$. The existence of such “wild” pro-vector bundles makes very likely the existence of pro-hermitian vector bundles satisfying (7.28).

7.3.3. Monotonicity properties. From now on, we will focus on the properties of the invariants \underline{h}_θ^0 and \overline{h}_θ^0 associated to pro-hermitian vector bundles and we shall leave the formulation of the dual properties of the invariants \underline{h}_θ^1 and \overline{h}_θ^1 associated to ind-hermitian vector bundles to the interested reader.

Proposition 7.3.5. *Let \widehat{E} and \widehat{F} be two pro-hermitian vector bundles over $\text{Spec } \mathcal{O}_K$, and let $\phi : \widehat{E} \rightarrow \widehat{F}$ be a morphism in $\text{Hom}_{\mathcal{O}_K}^{\leq 1}(\widehat{E}, \widehat{F})$.*

If $\hat{\phi} : \widehat{E} \rightarrow \widehat{F}$ is injective, then

$$(7.29) \quad \underline{h}_\theta^0(\widehat{E}) \leq \underline{h}_\theta^0(\widehat{F}).$$

If $\hat{\phi} : \widehat{E} \rightarrow \widehat{F}$ is injective and a strict morphism of topological \mathcal{O}_K -modules, then

$$(7.30) \quad \overline{h}_\theta^0(\widehat{E}) \leq \overline{h}_\theta^0(\widehat{F}).$$

If ϕ_K has a dense image²⁹, then

$$(7.31) \quad h_\theta^1(\widehat{E}) \geq h_\theta^1(\widehat{F}).$$

Proof. The inequality (7.29) when the morphism $\hat{\phi}$ is a straightforward consequences of the definitions of $\underline{h}_\theta^0(\widehat{E})$ and $\underline{h}_\theta^0(\widehat{F})$. Indeed for any (\widehat{E}', ι) in $\mathcal{L}(\widehat{E})$, the pair $(\widehat{E}', \phi \circ \iota)$ belongs to $\mathcal{L}(\widehat{F})$.

To complete the proof of the proposition, observe that, for any $V \in \mathcal{U}(\widehat{F})$, its inverse image $U := \phi^{-1}(V)$ is an element of $\mathcal{U}(\widehat{E})$, and there exists a unique morphism of \mathcal{O}_K -modules $\phi_V : E_U := \widehat{E}/U \rightarrow F_V := \widehat{F}/V$ such that the following diagram is commutative:

$$\begin{array}{ccc} \widehat{E} & \xrightarrow{\hat{\phi}} & \widehat{F} \\ p_U^{\widehat{E}} \downarrow & & \downarrow p_V^{\widehat{F}} \\ E_U & \xrightarrow{\phi_V} & F_V. \end{array}$$

Since ϕ belongs to $\text{Hom}_{\mathcal{O}_K}^{\leq 1}(\widehat{E}, \widehat{F})$, the morphism ϕ_V belongs to $\text{Hom}_{\mathcal{O}_K}^{\leq 1}(\overline{E}_U, \overline{F}_V)$. Moreover, by construction, it is injective. Consequently, according to Proposition 3.3.2, 1), we have:

$$(7.32) \quad h_\theta^0(\overline{E}_{\hat{\phi}^{-1}(V)}) \leq h_\theta^0(\overline{F}_V).$$

To prove (7.30), observe that, when $\hat{\phi} : \widehat{E} \rightarrow \widehat{F}$ is injective and strict, $\phi^{-1}(V)$ converges to 0 in \widehat{E} if V converges to 0 in \widehat{F} . Consequently:

$$(7.33) \quad \liminf_{U \in \mathcal{U}(\widehat{E})} h_\theta^0(\overline{E}_U) \leq \liminf_{V \in \mathcal{U}(\widehat{F})} h_\theta^0(\overline{E}_{\hat{\phi}^{-1}(V)}).$$

Then inequality (7.30) follows from (7.32) and (7.33).

Besides, when ϕ_K has a dense image, the K -linear map

$$\phi_{V,K} : E_{\hat{\phi}^{-1}(V),K} \rightarrow F_{V,K}$$

²⁹namely, if $\phi_K(\widehat{E}_K)$ is dense in $\widehat{F}_K \simeq \lim_{V \in \mathcal{U}(\widehat{F})} (F/V) \otimes_{\mathcal{O}_K} K$ equipped with its natural pro-discrete topology.

is surjective for every V in $\mathcal{U}(\overline{F})$. Consequently, according to Proposition 3.3.2, 2), we have:

$$(7.34) \quad h_{\theta}^1(\overline{E}_{\hat{\phi}^{-1}(V)}) \geq h_{\theta}^1(\overline{F}_V).$$

The inequality follows by taking the supremum over V in $\mathcal{U}(\hat{F})$, thanks to the expression (7.5) of h_{θ}^1 as a supremum. \square

7.4. Examples.

7.4.1. *Countable products of hermitian line bundles over $\text{Spec } \mathbb{Z}$.* Let $\lambda := (\lambda_i)_{i \in \mathbb{N}}$ be an element of $\mathbb{R}_+^{*\mathbb{N}}$.

For any integer $n \in \mathbb{N}$, we may consider the n -tuple

$$\lambda_{<n} := (\lambda_i)_{0 \leq i < n}$$

in \mathbb{R}_+^n and the associated hermitian vector bundle $\overline{V}_{\lambda_{<n}}$ of rank n over $\text{Spec } \mathbb{Z}$, as defined in 3.5 above. The morphisms

$$q_n : \overline{V}_{\lambda_{<n+1}} \longrightarrow \overline{V}_{\lambda_{<n}}$$

defined by

$$q_n(x_0, \dots, x_{n-1}, x_n) := (x_0, \dots, x_{n-1})$$

are surjective admissible, and we may consider the projective limit

$$\hat{\overline{V}}_{\lambda} := \varprojlim_n \overline{V}_{\lambda_{<n}}$$

of the projective system:

$$\overline{V}_{\lambda_{<0}} \xleftarrow{q_0} \overline{V}_{\lambda_{<1}} \xleftarrow{q_1} \dots \xleftarrow{q_{n-1}} \overline{V}_{\lambda_{<n}} \xleftarrow{q_n} \overline{V}_{\lambda_{<n+1}} \xleftarrow{q_{n+1}} \dots$$

The underlying topological \mathbb{Z} -module \hat{V}_{λ} of the pro-hermitian vector bundle $\hat{\overline{V}}_{\lambda}$ may be identified with $\mathbb{Z}^{\mathbb{N}}$, equipped with the product topology of the discrete topology on each factor \mathbb{Z} . The locally convex complex vector space $\hat{V}_{\lambda, \mathbb{C}}$ may be identified with $\mathbb{C}^{\mathbb{N}}$, also equipped with its natural product topology, and the Hilbert space $V_{\lambda, \mathbb{C}}^{\text{Hilb}}$ with the subspace of $\mathbb{C}^{\mathbb{N}}$ consisting in the elements $(x_i)_{i \in \mathbb{N}}$ such that $\sum_{i \in \mathbb{N}} \lambda_i |x_i|^2 < +\infty$. Moreover, for any element $(x_i)_{i \in \mathbb{N}}$ of $V_{\lambda, \mathbb{C}}^{\text{Hilb}}$,

$$\|(x_i)_{i \in \mathbb{N}}\|_{\hat{V}_{\lambda, \mathbb{C}}}^2 = \sum_{i \in \mathbb{N}} \lambda_i |x_i|^2.$$

For any $n \in \mathbb{N}$, we may also consider the injective isometric morphism

$$i_n : \overline{V}_{\lambda_{<n}} \longrightarrow \hat{\overline{V}}_{\lambda}$$

defined by:

$$i_n(x_0, \dots, x_{n-1}) := (x_0, \dots, x_{n-1}, 0, 0, \dots).$$

Proposition 7.4.1. *With the above notation, the following equalities hold in $[0, +\infty]$:*

$$(7.35) \quad \underline{h}_{\theta}^0(\hat{\overline{V}}_{\lambda}) = \overline{h}_{\theta}^0(\hat{\overline{V}}_{\lambda}) = \sum_{i \in \mathbb{N}} \tau(\lambda_i)$$

and

$$(7.36) \quad h_{\theta}^1(\hat{\overline{V}}_{\lambda}) = \sum_{i \in \mathbb{N}} \tau(\lambda_i^{-1}).$$

Proof. As shown in Section 3.5, we have:

$$h_\theta^0(\tilde{V}_{\lambda_{<n}}) = \sum_{0 \leq i < n} \tau(\lambda_i) \text{ and } h_\theta^1(\tilde{V}_{\lambda_{<n}}) = \sum_{0 \leq i < n} \tau(\lambda_i^{-1}).$$

Besides, the existence of the isometric injections i_n shows that, for every $n \in \mathbb{N}$,

$$h_\theta^0(\tilde{V}_{\lambda_{<n}}) \leq \underline{h}_\theta^0(\hat{V}_\lambda),$$

and, by the very definition of $\bar{h}_\theta^0(\tilde{V}_\lambda)$,

$$\bar{h}_\theta^0(\tilde{V}_\lambda) \leq \liminf_{n \rightarrow +\infty} h_\theta^0(\tilde{V}_{\lambda_{<n}}).$$

Together with the inequality $\underline{h}_\theta^0(\tilde{V}_\lambda) \leq \bar{h}_\theta^0(\tilde{V}_\lambda)$, this implies (7.35).

Formula (7.36) follows from (7.4) applied to the projective system $(\tilde{V}_{\lambda_{<\bullet}})$. \square

Observe that, as a straightforward consequence of (7.35) and (7.36) and of the asymptotic behavior (3.27) of the function τ , we obtain the finiteness conditions:

$$(7.37) \quad \underline{h}_\theta^0(\hat{V}_\lambda) < +\infty \iff \bar{h}_\theta^0(\hat{V}_\lambda) < +\infty \iff \sum_{i \in \mathbb{N}} e^{-\pi \lambda_i} < +\infty$$

and

$$(7.38) \quad h_\theta^1(\hat{V}_\lambda) < +\infty \iff \sum_{i \in \mathbb{N}} e^{-\pi / \lambda_i} < +\infty.$$

The content of Proposition 7.4.1 may be reformulated as follows:

Corollary 7.4.2. *For any countable family $(\bar{L}_i)_{i \in \mathbb{N}}$ of hermitian line bundles over $\text{Spec } \mathbb{Z}$, the θ -invariants of the pro-hermitian vector*

$$\bigoplus_{i \in \mathbb{N}} \bar{L}_i := \varprojlim_n \bigoplus_{0 \leq i \leq n} \bar{L}_i,$$

satisfy:

$$\underline{h}_\theta^0(\bigoplus_{i \in \mathbb{N}} \bar{L}_i) = \bar{h}_\theta^0(\bigoplus_{i \in \mathbb{N}} \bar{L}_i) = \sum_{i \in \mathbb{N}} \tau(e^{-2\widehat{\deg} \bar{L}_i}) = \sum_{i \in \mathbb{N}} (\widehat{\deg}^+ \bar{L}_i + \eta(\widehat{\deg} \bar{L}_i))$$

and

$$h_\theta^1(\bigoplus_{i \in \mathbb{N}} \bar{L}_i) = \sum_{i \in \mathbb{N}} \tau(e^{2\widehat{\deg} \bar{L}_i}) = \sum_{i \in \mathbb{N}} (\widehat{\deg}^- \bar{L}_i + \eta(\widehat{\deg} \bar{L}_i)).$$

\square

7.4.2. Pro-euclidean lattices defined by formal series and holomorphic functions on disks. The computation of the θ -invariants of countable products of one dimensional euclidean lattices in the previous paragraph, although quite elementary, allows one to investigate the θ -invariants attached to the arithmetic avatars of the Hardy and Bergman spaces introduced in paragraph 6.6.

Proposition 7.4.3. *For every $(R, \delta) \in \mathbb{R}_+^* \times \mathbb{R}$, the following equalities hold in $[0, +\infty]$:*

$$(7.39) \quad \underline{h}_\theta^0(\widehat{H}(R) \otimes \overline{\mathcal{O}}(\delta)) = \bar{h}_\theta^0(\widehat{H}(R) \otimes \overline{\mathcal{O}}(\delta)) = \sum_{n \in \mathbb{N}} \tau(R^{2n} e^{-2\delta}).$$

Moreover

$$h(R, \delta) := \sum_{n \in \mathbb{N}} \tau(R^{2n} e^{-2\delta})$$

is finite if and only if $R > 1$, and, for any $R > 1$, we have:

$$(7.40) \quad h(R, \delta) = \frac{1}{2 \log R} \delta^2 + O(\delta) \quad \text{when } \delta \longrightarrow +\infty.$$

Similar results hold concerning the Bergman pro-euclidean lattices $\widehat{B}(R)$, and are left as exercises for the reader.

Proof. The isomorphism in $CTC_{\mathbb{Z}}$

$$\begin{aligned} \mathbb{Z}^{\mathbb{N}} &\xrightarrow{\sim} \mathbb{Z}[[X]] \\ (a_n)_{n \in \mathbb{N}} &\longmapsto \sum_{n \in \mathbb{N}} a_n X^n. \end{aligned}$$

“extends” to an isomorphism of pro-hermitian vector bundles over $\text{Spec } \mathbb{Z}$:

$$V_{\lambda_H(R, \delta)} \xrightarrow{\sim} \widehat{H}(R) \otimes \overline{\mathcal{O}}(\delta),$$

where

$$\lambda_H(R, \delta) := (R^{2n} e^{-2\delta})_{n \in \mathbb{N}}.$$

The relations (7.39) therefore follows from Proposition 7.4.1, equation (7.35).

As we discussed in Paragraph (3.5), for any $x \in \mathbb{R}_+^*$, we may express $\tau(x)$ as the sum

$$\tau(x) = \left(-\frac{1}{2} \log x\right)^+ + \eta\left(-\frac{1}{2} \log x\right),$$

where η denotes an even positive continuous function on \mathbb{R} which satisfies

$$\eta(t) \leq 3e^{-\pi e^{2|t|}} \quad \text{for any } t \in \mathbb{R},$$

or equivalently,

$$\eta\left(-\frac{1}{2} \log x\right) \leq 3e^{-\pi \max(|x|, |x|^{-1})} \quad \text{for any } x \in \mathbb{R}_+^*.$$

Consequently we have:

$$h(R, \delta) = \sum_{n \in \mathbb{N}} (-n \log R + \delta)^+ + \sum_{n \in \mathbb{N}} \eta(-n \log R + \delta).$$

When $-\log R$ is positive (rest. vanishes), the first (rest. second) sum is $+\infty$. Therefore $h(R, \delta) = +\infty$ when $R \in]0, 1]$.

When $\log R$ is positive both sums are clearly finite. Actually, we have:

$$\sum_{n \in \mathbb{N}} \eta(-n \log R + \delta) \leq \sum_{n \in \mathbb{Z}} \eta(-n \log R + \delta) \leq 6 \sum_{n \in \mathbb{N}} e^{-\pi R^n} < +\infty.$$

Moreover, when δ goes to $+\infty$,

$$\sum_{n \in \mathbb{N}} (-n \log R + \delta)^+ = \sum_{\substack{n \in \mathbb{N} \\ n \leq \delta / \log R}} (-n \log R + \delta) = \frac{1}{2 \log R} \delta^2 + O(\delta).$$

This completes the proof of (7.40). □

7.4.3. *A pro-hermitian vector bundle of infinite rank with vanishing \overline{h}_θ^0 .* For any positive integer a , we may consider the constant a -tuple $a^{\times a} := (a, \dots, a)$ and, with the notation of Section 3.5, the hermitian vector bundle

$$\overline{E}_a := \overline{V}_{a^{\times a}},$$

namely the hermitian vector bundle over $\mathrm{Spec} \mathbb{Z}$ defined by the lattice \mathbb{Z}^a inside \mathbb{R}^a equipped with the euclidean norm

$$\|\cdot\|_{\overline{E}_a} := \|\cdot\|_{a^{\times a}} : (x_i)_{1 \leq i \leq a} \mapsto (a \sum_{1 \leq i \leq a} x_i^2)^{1/2}.$$

We leave the proof of the next lemma as an exercise for the reader.

Lemma 7.4.4. *For any two positive integers a and b such that $a \mid b$, one defines an admissible surjective morphism of hermitian vector bundles over $\mathrm{Spec} \mathbb{Z}$,*

$$p_{ab} : \overline{E}_b \mapsto \overline{E}_a,$$

by letting

$$p_{ab}(x_i)_{1 \leq i \leq b} = (y_j)_{1 \leq j \leq a}, \text{ where } y_j := \sum_{i=1+(j-1)d}^{jd} x_i.$$

Moreover, for any three positive integers a , b , and c such that $a \mid b$ and $b \mid c$, the following transitivity relation holds:

$$p_{ab} \circ p_{bc} = p_{ac}.$$

□

Observe also that

$$h_\theta^0(\overline{E}_a) = a\tau(a) \sim 2ae^{-\pi a} \text{ when } a \mapsto +\infty.$$

Consequently, for any infinite set \mathcal{A} of positive integers in which the divisibility relation \mid is filtering, we may consider the pro-hermitian vector bundle (of infinite rank) over $\mathrm{Spec} \mathbb{Z}$:

$$\tilde{E}_{\mathcal{A}} := \varprojlim_{(\mathcal{A}, \mid)} \overline{E}_a.$$

Since $\lim_{(\mathcal{A}, \mid)} h_\theta^0(\overline{E}_a) = 0$, it satisfies $\overline{h}_\theta^0(\tilde{E}_{\mathcal{A}}) = 0$.

8. SUMMABLE PROJECTIVE SYSTEMS OF HERMITIAN VECTOR BUNDLES AND FINETENESS OF θ -INVARIANTS

We still denote by K a number field, by \mathcal{O}_K its ring of integers and by π the morphism of schemes from $\mathrm{Spec} \mathcal{O}_K$ to $\mathrm{Spec} \mathbb{Z}$.

8.1. Main theorem. This section is mainly devoted to a proof of the following theorem, which shall play a key role in the sequel of this article for constructing pro-hermitian vector bundles with finite and well-behaved invariants $\underline{h}_\theta^0(\widehat{E})$ and $\overline{h}_\theta^0(\widehat{E})$.

Theorem 8.1.1. *Let*

$$\overline{E}_\bullet : \overline{E}_0 \xleftarrow{q_0} \overline{E}_1 \xleftarrow{q_1} \dots \xleftarrow{q_{i-1}} \overline{E}_i \xleftarrow{q_i} \overline{E}_{i+1} \xleftarrow{q_{i+1}} \dots$$

be a projective system of surjective admissible morphisms of hermitian vector bundles over $\mathrm{Spec} \mathcal{O}_K$, and consider the associated pro-hermitian vector bundle over $\mathrm{Spec} \mathcal{O}_K$:

$$\widehat{E} := \varprojlim_i \overline{E}_i.$$

For every $i \in \mathbb{N}$, let us denote by $\overline{\ker q_i}$ the hermitian vector bundle over $\mathrm{Spec} \mathcal{O}_K$ defined as the kernel of q_i equipped with the hermitian structure induced by the one of \overline{E}_{i+1} .

If there exists $\epsilon \in \mathbb{R}_+^*$ such that

$$(8.1) \quad \sum_{i \in \mathbb{N}} h_\theta^0(\overline{\ker q_i} \otimes_{\mathcal{O}_K} \overline{\mathcal{O}_{\text{Spec } \mathcal{O}_K}(\epsilon)}) < +\infty,$$

then the limit $\lim_{i \rightarrow +\infty} h_\theta^0(\overline{E}_i)$ exists in \mathbb{R}_+ and

$$(8.2) \quad \underline{h}_\theta^0(\widehat{E}) = \overline{h}_\theta^0(\widehat{E}) = \overline{h}_\theta^0(\pi_* \widehat{E}) = \lim_{i \rightarrow +\infty} h_\theta^0(\overline{E}_i).$$

Moreover, for any $k \in \mathbb{N}$, the non-negative real number (8.2) admits the following upper bound:

$$(8.3) \quad \lim_{i \rightarrow +\infty} h_\theta^0(\overline{E}_i) \leq h_\theta^0(\overline{E}_k) + \sum_{j=k}^{+\infty} h_\theta^0(\overline{\ker q_j}).$$

When $\text{Spec } \mathcal{O}_K = \text{Spec } \mathbb{Z}$, this theorem will be established in a more precise form as Proposition 8.3.1 and Theorem 8.3.4, 2), below. The simple derivation of Theorem 8.1.1 for an arbitrary number field K from its special case where $\text{Spec } \mathcal{O}_K = \text{Spec } \mathbb{Z}$ is presented in Section 8.3.3.

The proof of Proposition 8.3.1 and of Theorem 8.3.4 are presented in the next subsections (Sections 8.2 to 8.5) of this section.

Section 8.6 will be devoted to establishing that Theorem 8.1.1 is basically optimal : we will show that any pro-hermitian vector bundle \widehat{E} such that $\underline{h}_\theta^0(\widehat{E}) = \overline{h}_\theta^0(\widehat{E}) < +\infty$ may be realized as the projective limit $\varprojlim_i \overline{E}_i$ of a projective system of hermitian vector bundles \overline{E}_\bullet satisfying a summability condition similar to (8.1). This additional result will finally allow us to give sensible definitions of *strongly summable* and of θ -*finite* pro-hermitian vector bundles in Section 8.7.

The proofs in this part will use some basic facts concerning measure theory on Polish spaces constructed as projective limits of countable systems of countable discrete sets. These facts are presented in a form suited to our need in Appendix C.

8.2. Preliminaries. We begin by introducing the notation and the definitions that will be used in the formulation and in the proofs of Proposition 8.3.1 and Theorem 8.3.4 in Sections 8.3–8.5.

8.2.1. *Notation.* Let

$$\overline{E}_\bullet : \overline{E}_0 \xleftarrow{q_0} \overline{E}_1 \xleftarrow{q_1} \dots \xleftarrow{q_{i-1}} \overline{E}_i \xleftarrow{q_i} \overline{E}_{i+1} \xleftarrow{q_{i+1}} \dots$$

be a projective system of surjective admissible morphisms of hermitian vector bundles over $\text{Spec } \mathbb{Z}$, and let

$$\tilde{E} := \varprojlim_i \overline{E}_i = (\widehat{E}, E_{\mathbb{R}}^{\text{Hilb}}, \|\cdot\|)$$

be the associated pro-hermitian vector bundle over $\text{Spec } \mathbb{Z}$.

For every $(i, j) \in \mathbb{N}^2$ such that $i \leq j$, we let:

$$p_{ij} := q_i \circ q_{i+1} \circ \dots \circ q_{j-1} : E_j \longrightarrow E_i,$$

and we denote by

$$p_i : \widehat{E} := \varprojlim_k E_k \longrightarrow E_i$$

the i -th projection morphism, and by

$$U_i := \ker p_i$$

its kernel.

In this section, we shall apply the basic results concerning measures on countable sets and on their projective limits recalled in Appendix C to the projective system of countable sets

$$E_\bullet : E_0 \xleftarrow{q_0} E_1 \xleftarrow{q_1} \dots \xleftarrow{q_{i-1}} E_i \xleftarrow{q_i} E_{i+1} \xleftarrow{q_{i+1}} \dots$$

underlying \overline{E}_\bullet . A key point will be that, to the hermitian vector bundle \overline{E}_i , is naturally attached a bounded positive measure of total mass $\exp h_\theta^0(\overline{E}_i)$ on the underlying set E_i .

Indeed, to any (finite dimensional) hermitian vector bundle $\overline{V} := (V, \|\cdot\|)$ over $\text{Spec } \mathbb{Z}$, we may attach the positive measure on the countable set V :

$$\gamma_{\overline{V}} := e^{-\pi \|\cdot\|_{\overline{V}}^2} \sum_{v \in V} \delta_v = \sum_{v \in V} e^{-\pi \|v\|_{\overline{V}}^2} \delta_v.$$

This measure, which occurs implicitly in the arguments *à la* Banaszczyk of paragraph 4.1.2 and section 4.1.2, has a finite total mass:

$$\gamma_{\overline{V}}(V) = \sum_{v \in V} e^{-\pi \|v\|_{\overline{V}}^2} = \exp(h_\theta^0(\overline{V})).$$

For every $i \in \mathbb{N}$, we may consider the continuous function $\|p_i\|_{\overline{E}_i}$ on $\widehat{E}_{\mathbb{R}}$. We thus define a non-decreasing sequence of functions on $\widehat{E}_{\mathbb{R}}$, which converges (pointwise) towards the function $\|\cdot\|$ on $\widehat{E}_{\mathbb{R}}$ defined as the Hilbert norm on $E_{\mathbb{R}}^{\text{Hilb}}$ and as $+\infty$ on $\widehat{E}_{\mathbb{R}} \setminus E_{\mathbb{R}}^{\text{Hilb}}$. In particular,

$$\|\cdot\| : \widehat{E}_{\mathbb{R}} \longrightarrow [0, +\infty]$$

is lower semi-continuous.

For any $\eta \in \mathbb{R}_+^*$, we may also consider the functions

$$f_{i,\eta} := \exp(-\eta \pi \|p_i\|_{\overline{E}_i}^2)$$

and

$$f_\eta := \exp(-\eta \pi \|\cdot\|^2)$$

on $\widehat{E}_{\mathbb{R}}$. Then $(f_{i,\eta})_{i \in \mathbb{N}}$ is a non-increasing sequence of continuous functions from $\widehat{E}_{\mathbb{R}}$ to $[0, 1]$ and converges pointwise to f_η , which is therefore upper semi-continuous.

Observe that, for any $x \in \widehat{E}_{\mathbb{R}}$,

$$f_\eta(x) = \begin{cases} e^{-\eta \pi \|x\|^2} & \text{if } x \in E_{\mathbb{R}}^{\text{Hilb}} \\ 0 & \text{if } x \in \widehat{E}_{\mathbb{R}} \setminus E_{\mathbb{R}}^{\text{Hilb}} \end{cases}$$

and that consequently:

$$(8.4) \quad \lim_{\eta \rightarrow 0_+} f_\eta(x) = 1_{E_{\mathbb{R}}^{\text{Hilb}}}(x).$$

8.2.2. Definitions. For every $i \in \mathbb{N}$, we consider the hermitian vector bundle $\overline{\ker q_i}$, defined as the kernel of q_i equipped with the hermitian structure induced by the one of \overline{E}_{i+1} . By construction, its fits into an admissible exact sequence

$$(8.5) \quad \overline{S}_i : 0 \longrightarrow \overline{\ker q_i} \xrightarrow{\quad} \overline{E}_{i+1} \xrightarrow{q_i} \overline{E}_i \longrightarrow 0.$$

We shall say that the projective system \overline{E}_\bullet is *summable* when it satisfies the following condition:

$$\mathbf{Sum}(\overline{E}_\bullet) : \sum_{i \in \mathbb{N}} h_\theta^0(\overline{\ker q_i}) < +\infty.$$

For any $\lambda \in \mathbb{R}$, we may “twist” \overline{E}_\bullet by the hermitian line bundle $\overline{\mathcal{O}}(\lambda)$ and consider the attached summability condition:

$$\mathbf{Sum}(\overline{E}_\bullet \otimes \overline{\mathcal{O}}(\lambda)) : \sum_{i \in \mathbb{N}} h_\theta^0(\overline{\ker q_i} \otimes \overline{\mathcal{O}}(\lambda)) < +\infty.$$

Observe that, when this condition is satisfied for some value λ_0 of λ , then it is also satisfied for any λ in $] -\infty, \lambda_0]$. We shall say that \overline{E}_\bullet is *strongly summable* when $\mathbf{Sum}(\overline{E}_\bullet \otimes \overline{\mathcal{O}}(\epsilon))$ is satisfied for some $\epsilon \in \mathbb{R}_+^*$.

For any $i \in \mathbb{N}$ and any $t \in \mathbb{R}_+^*$, we may consider

$$\log \sum_{v \in E_i} e^{-\pi t \|v\|_{\overline{E}_i}^2} = h_\theta^0(\overline{E}_i \otimes \overline{\mathcal{O}}(-(\log t)/2)).$$

As a function of t , it is convex on \mathbb{R}_+^* . We shall say that $M_{\overline{E}_\bullet}(t)$ is defined when the limit

$$M_{\overline{E}_\bullet}(t) = \lim_{i \rightarrow +\infty} \log \sum_{v \in E_i} e^{-\pi t \|v\|_{\overline{E}_i}^2} = \lim_{i \rightarrow +\infty} h_\theta^0(\overline{E}_i \otimes \overline{\mathcal{O}}(-(\log t)/2))$$

exists in \mathbb{R}_+ .

8.2.3. Convexity of $M_{\overline{E}_\bullet}$. The following lemma, which is a straightforward consequence of the previous observations, turns out to play a key role in our derivation of Theorem 8.1.1 (through the proof of Corollary 8.3.3 *infra*).

Lemma 8.2.1. *If $M_{\overline{E}_\bullet}(t)$ is defined for every t in some compact interval $[a, b]$ in \mathbb{R}_+^* , then the function*

$$M_{\overline{E}_\bullet} : [a, b] \longrightarrow \mathbb{R}_+$$

is convex and non-increasing. In particular it is continuous on $]a, b]$. \square

8.3. Summable projective systems of hermitian vector bundles and associated measures.

8.3.1. Existence of the limit $\lim_{i \rightarrow +\infty} h_\theta^0(\overline{E}_i)$.

Proposition 8.3.1. *If the projective system \overline{E}_\bullet is summable, then the limit*

$$(8.6) \quad \lim_{i \rightarrow +\infty} h_\theta^0(\overline{E}_i)$$

exists in $[0, +\infty[$.

Moreover, for any $k \in \mathbb{N}$,

$$\lim_{i \rightarrow +\infty} h_\theta^0(\overline{E}_i) \leq h_\theta^0(\overline{E}_k) + \sum_{j=k}^{+\infty} h_\theta^0(\overline{\ker q_j}).$$

Proof. The subadditivity of h_θ^0 (Proposition 3.8.1) applied to the admissible short exact sequence $\overline{\mathcal{S}}_i$ (see (8.5)) shows that, for any $i \in \mathbb{N}$,

$$(8.7) \quad h_\theta^0(\overline{E}_{i+1}) \leq h_\theta^0(\overline{E}_i) + h_\theta^0(\overline{\ker q_i}).$$

The sequence

$$(h_\theta^0(\overline{E}_i) - \sum_{0 \leq j < i} h_\theta^0(\overline{\ker q_j}))_{i \in \mathbb{N}}$$

is therefore non-increasing and bounded below by $-\Sigma$, where $\Sigma := \sum_{0 \leq j < +\infty} h_\theta^0(\overline{\ker q_j})$, and consequently admits a limit l in $[-\Sigma, +\infty[$. This proves that $(h_\theta^0(\overline{E}_i))_{i \in \mathbb{N}}$ converges to $l + \Sigma \in \mathbb{R}^+$.

Moreover, for any $k \in \mathbb{N}$,

$$l + \Sigma \leq (h_\theta^0(\overline{E}_k) - \sum_{0 \leq j < k} h_\theta^0(\overline{\ker q_j})) + \Sigma = h_\theta^0(\overline{E}_k) + \sum_{k \leq j < +\infty} h_\theta^0(\overline{\ker q_j}).$$

\square

Corollary 8.3.2. *If the projective system \overline{E}_\bullet is summable (resp. strongly summable), then $M_{\overline{E}_\bullet}(t)$ is defined for any t in $[1, +\infty[$ (resp. for any t in some open interval containing $[1, +\infty[$).* \square

Proof. Observe that the validity of $\mathbf{Sum}(\overline{E}_\bullet)$ implies the one of $\mathbf{Sum}(\overline{E}_\bullet \otimes \overline{\mathcal{O}}(\lambda))$ for every λ in \mathbb{R}_- and that, for any $(t, \lambda) \in \mathbb{R}_+^* \times \mathbb{R}$, $M_{\overline{E}_\bullet \otimes \overline{\mathcal{O}}(\lambda)}(t)$ and $M_{\overline{E}_\bullet}(e^{-2\lambda}t)$ are defined simultaneously and are then equal.

Therefore to establish the Corollary, it is enough to show that $M_{\overline{E}_\bullet}(1)$ is defined when \overline{E}_\bullet is summable: this is precisely the existence of the limit (8.6) established in Proposition 8.3.1. \square

Taking Lemma 8.2.1 into account, we finally obtain:

Corollary 8.3.3. *When \overline{E}_\bullet is strongly summable, the convex function $M_{\overline{E}_\bullet}$ on $[1, +\infty[$ is continuous at 1.* \square

8.3.2. *The main technical result.* We may now formulate the main technical result of this section. Its proof will be the object of the next two Sections, and will notably rely on the results concerning measures on projective limits on countable sets established in Appendix C.

Theorem 8.3.4. 1) *If the projective system \overline{E}_\bullet is summable, then the pro-hermitian vector bundle $\widehat{E} := \varprojlim_i \overline{E}_i$ satisfies:*

$$(8.8) \quad \overline{h}_\theta^0(\widehat{E}) = \lim_{i \rightarrow +\infty} h_\theta^0(\overline{E}_i).$$

Moreover, for any $i \in \mathbb{N}$, the sequence $(p_{ij*} \gamma_{\overline{E}_j})_{j \in \mathbb{N}_{\geq i}}$ converges to some limit μ_i in $\mathcal{M}_+^b(E_i)$, and there exists a unique measure $\mu_{\overline{E}_\bullet}$ in $\mathcal{M}_+^b(\widehat{E})$ such that, for any $i \in \mathbb{N}$,

$$\mu_i = p_{i*} \mu_{\overline{E}_\bullet}.$$

The mass of the measure $\mu_{\overline{E}_\bullet}$ is ≥ 1 and satisfies:

$$(8.9) \quad \overline{h}_\theta^0(\widehat{E}) = \log \mu_{\overline{E}_\bullet}(\widehat{E}).$$

Moreover, for any $v \in \widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}$, we have:

$$(8.10) \quad \mu_{\overline{E}_\bullet}(\{v\}) = e^{-\pi \|v\|^2}.$$

2) *If the projective system \overline{E}_\bullet is summable and if the convex function $M_{\overline{E}_\bullet} : [1, +\infty[\rightarrow \mathbb{R}_+$ is continuous at 1 — for instance, if \overline{E}_\bullet is strongly summable — then:*

$$(8.11) \quad \mu_{\overline{E}_\bullet} = \sum_{v \in \widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}} e^{-\pi \|v\|^2} \delta_v$$

and

$$(8.12) \quad \overline{h}_\theta^0(\widehat{E}) = \underline{h}_\theta^0(\widehat{E}) = \log \sum_{v \in \widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}} e^{-\pi \|v\|^2}.$$

Observe that, for any pro-hermitian vector bundle \widehat{E} such that $\underline{h}_\theta^0(\widehat{E}) < +\infty$, the right-hand side of (8.11) defines a measure in $\mathcal{M}_+^b(\widehat{E})$, that we shall denote by $\mu_{\widehat{E}}$. It satisfies

$$\underline{h}_\theta^0(\widehat{E}) = \log \mu_{\widehat{E}}(\widehat{E}).$$

and the conclusion of Theorem 8.3.4, 2), may be rephrased as the equality:

$$\mu_{\overline{E}_\bullet} = \mu_{\widehat{E}}.$$

8.3.3. *Completion of the proof of Theorem 8.1.1.* Before we proceed to the proof of Theorem 8.3.4, we explain how, taking Proposition 8.3.1 and Theorem 8.3.4, 2), for granted, one easily establishes Theorem 8.1.1.

Let us use the notation of Theorem 8.1.1. From Theorem 8.3.4, 2), applied to the direct images $\pi_* \overline{E}_\bullet$ and $\pi_* \widehat{\overline{E}} \simeq \varprojlim_i \pi_* \overline{E}_i$ of the projective system \overline{E}_\bullet and of its limit $\widehat{\overline{E}}$ by the morphism $\pi : \text{Spec } \mathcal{O}_K \rightarrow \text{Spec } \mathbb{Z}$, we obtain the existence of the limit $\lim_{i \rightarrow +\infty} h_\theta^0(\overline{E}_i)$ and the relations:

$$(8.13) \quad \underline{h}_\theta^0(\widehat{\overline{E}}) = \overline{h}_\theta^0(\pi_* \widehat{\overline{E}}) = \lim_{i \rightarrow +\infty} h_\theta^0(\widehat{\overline{E}}_i).$$

(Indeed, according to (3.12) and (7.14), we have:

$$\underline{h}_\theta^0(\pi_* \overline{E}_i) = \underline{h}_\theta^0(\overline{E}_i), \quad h_\theta^0(\pi_* \overline{\ker q_i} \otimes_{\mathbb{Z}} \overline{\mathcal{O}}(\epsilon)) = h_\theta^0(\overline{\ker q_i} \otimes_{\mathcal{O}_K} \overline{\mathcal{O}_{\text{Spec } \mathcal{O}_K}(\epsilon)}),$$

and

$$\underline{h}_\theta^0(\pi_* \widehat{\overline{E}}) = \underline{h}_\theta^0(\widehat{\overline{E}}).$$

Moreover, the inequality (7.16) and the very definition of $\overline{h}_\theta^0(\widehat{\overline{E}})$ show that:

$$\overline{h}_\theta^0(\pi_* \widehat{\overline{E}}) \leq \overline{h}_\theta^0(\widehat{\overline{E}}) \leq \lim_{i \rightarrow +\infty} h_\theta^0(\overline{E}_i).$$

Combined with (8.13), this completes the proof of (8.2).

8.4. **Proof of Theorem 8.3.4 – I. The equality $\overline{h}_\theta^0(\widehat{\overline{E}}) = \lim_{i \rightarrow +\infty} h_\theta^0(\overline{E}_i)$.** In this paragraph, we consider a *summable* projective system (\overline{E}_\bullet) of surjective admissible morphisms of hermitian vector bundles over $\text{Spec } \mathbb{Z}$. According to Proposition 8.3.1, the limit

$$l := \lim_{i \rightarrow +\infty} h_\theta^0(\overline{E}_i)$$

exists in \mathbb{R}_+ . Clearly,

$$\liminf_{U \in \mathcal{U}(\widehat{\overline{E}})} h_\theta^0(\overline{E}_U) \leq \lim_{i \rightarrow +\infty} h_\theta^0(\overline{E}_i).$$

Therefore, to prove (8.8), it is enough to show that

$$(8.14) \quad \liminf_{U \in \mathcal{U}(\widehat{\overline{E}})} h_\theta^0(\overline{E}_U) \geq l.$$

To achieve this, observe that, for any open saturated submodule U of $\widehat{\overline{E}}$, we may perform the following construction.

If i_0 is a large enough positive integer (depending on U), then for any $i \geq i_0$, the submodule U contains U_i , and we may consider the quotient map:

$$p_{U, U_i} : E_i \simeq \widehat{\overline{E}}/U_i \longrightarrow E_U := \widehat{\overline{E}}/U.$$

Its kernel defines a hermitian vector bundle \overline{K}_i^U which fits into an admissible short exact sequence

$$(8.15) \quad 0 \longrightarrow \overline{K}_i^U \longrightarrow \overline{E}_i \longrightarrow \overline{E}_U \longrightarrow 0.$$

Moreover, the morphism $q_i := p_{U, U_{i+1}} : E_{i+1} \longrightarrow E_i$ defines by restriction an admissible surjective morphism

$$q_i^U : \overline{K}_{i+1}^U \longrightarrow \overline{K}_i^U,$$

of kernel

$$\ker q_i^U = \ker q_i.$$

Moreover the natural euclidian structures on $\ker q_i^U$ and $\ker q_i$ (defined by the euclidean structures on \overline{K}_{i+1}^U and on \overline{E}_{i+1}) clearly coincide:

$$\overline{\ker q_i^U} = \overline{\ker q_i} \text{ for every } i \geq i_0.$$

Thus we may consider the summable projective system

$$\overline{K}_\bullet^U : \overline{K}_{i_0}^U \xleftarrow{q_{i_0}} \overline{K}_{i_0+1}^U \xleftarrow{q_{i_0+1}} \dots \xleftarrow{q_{i-1}} \overline{K}_i^U \xleftarrow{q_i} \overline{K}_{i+1}^U \xleftarrow{q_{i+1}} \dots$$

(The pro-hermitian vector bundle $\tilde{K}^U := \varprojlim_i \overline{K}_i^U = (\hat{K}^U, K_{\mathbb{R}}^{U, \text{Hilb}}, \|\cdot\|_{\tilde{K}^U})$ may also be directly defined by

$$\hat{K}^U = U, \quad K^{U, \text{Hilb}_{\mathbb{R}}} := \ker p_{U, \mathbb{R}|E_{\mathbb{R}}^{\text{Hilb}}}, \quad \text{and } \|\cdot\|_{\tilde{K}^U} := \|\cdot\|_{K^{U, \text{Hilb}_{\mathbb{R}}}},$$

where we denote by $p_U : \hat{E} \rightarrow E_U$ the quotient map from \hat{E} onto \hat{E}/U and by $p_{U, \mathbb{R}} : \hat{E}_{\mathbb{R}} \rightarrow E_{U, \mathbb{R}}$ its \mathbb{R} -linear continuous extension.)

According to Proposition 8.3.1, the limit

$$l(U) := \lim_{i \rightarrow +\infty} h_\theta^0(\overline{K}_i^U)$$

exists in \mathbb{R}_+ and satisfies

$$(8.16) \quad l(U) \leq h_\theta^0(\overline{K}_{i_0}^U) + \sum_{i \geq i_0} h_\theta^0(\overline{\ker q_i}).$$

Lemma 8.4.1. 1) For any open saturated submodule U of \hat{E} , we have:

$$(8.17) \quad l \leq h_\theta^0(\overline{E}_U) + l(U).$$

2) For any two open saturated submodules U and U' of \hat{E} ,

$$(8.18) \quad U' \subset U \implies l(U') \leq l(U).$$

3) For any $k \in \mathbb{N}$,

$$(8.19) \quad l(U_k) \leq \sum_{i \geq k} h_\theta^0(\overline{S}_i).$$

4) We have :

$$\lim_{U \in \mathcal{U}(\hat{E})} l(U) = 0.$$

Proof. 1) The subadditivity of h_θ^0 applied to the admissible short exact sequence (8.15) establishes the inequality

$$h_\theta^0(\overline{E}_i) \leq h_\theta^0(\overline{K}_i^U) + h_\theta^0(\overline{E}_U).$$

This yields (8.17) by letting i go to infinity.

2) For any large enough integer i , we have $K_i^{U'} \subset K_i^U$, and consequently

$$h_\theta^0(\overline{K}_i^{U'}) \leq h_\theta^0(\overline{K}_i^U).$$

This yields (8.18) by letting i go to infinity.

3) In the above construction, when $U = U_k$, we may choose $i_0 = k$. Then $\overline{K}_{i_0}^U = 0$, and (8.19) is nothing but (8.16).

4) follows from 2) and 3). □

The estimate (8.14) directly follows from assertions 1) and 4) in Lemma 8.4.1.

8.5. Proof of Theorem 8.3.4 – II. Convergence of measures. As in the previous paragraph, we consider a summable projective system \overline{E}_\bullet of hermitian vector bundles over $\text{Spec } \mathbb{Z}$.

Lemma 8.5.1. *For any $i \in \mathbb{N}$, the measures $\gamma_{\overline{E}_i}$ and $q_{i*}\gamma_{\overline{E}_{i+1}}$ on E_i satisfy:*

$$q_{i*}\gamma_{\overline{E}_{i+1}} \leq e^{h_\theta^0(\overline{\ker q_i})} \gamma_{\overline{E}_i}.$$

Proof. This is the content of Lemma 3.8.2, applied to the admissible short exact sequence \overline{S}_i (defined in (8.5)). \square

The existence of the limit measures μ_i on E_i and of the existence and the unicity of the measure $\mu_{\overline{E}_\bullet}$ now follows from Proposition C.2.1 of the Appendix, applied to $D_i = E_i$, $\gamma_i = \gamma_{\overline{E}_i}$, and $\lambda_i = h_\theta^0(\overline{E}_i)$. Indeed, according to the summability assumption on \overline{E}_\bullet , the sequence $(h_\theta^0(\overline{E}_i))_{i \in \mathbb{N}}$ belongs to $l^1(\mathbb{N})$.

According to (8.8), the equality (8.9) may also be written

$$\mu_{\overline{E}_\bullet}(\widehat{E}) = \lim_{i \rightarrow +\infty} e^{h_\theta^0(\overline{E}_i)}.$$

As $e^{h_\theta^0(\overline{E}_i)} = \gamma_{\overline{E}_i}(E_i)$, it is nothing but a reformulation of (C.6).

The equality (8.10) will follow from the following:

Lemma 8.5.2. *Let v be an element of $\widehat{E} \cap E_\mathbb{R}^{\text{Hilb}}$, and, for any $i \in \mathbb{N}$, let $v_i := p_i(v)$ be its image in E_i . Then, for any $i \in \mathbb{N}$, we have*

$$(8.20) \quad e^{-\pi\|v\|^2} \leq \mu_i(\{v_i\}) \leq e^{-\pi\|v_i\|_{\overline{E}_i}^2 + \sum_{k \geq i} h_\theta^0(\overline{\ker q_i})}.$$

Indeed, with the notation of Lemma 8.5.2, the set $\{v\}$ may be described as the countable decreasing intersection

$$\{v\} = \bigcap_{i \in \mathbb{N}} p_i^{-1}(v_i),$$

and therefore:

$$\mu_{\overline{E}_\bullet}(\{v\}) = \lim_{i \rightarrow +\infty} \mu_{\overline{E}_\bullet}(p_i^{-1}(v_i)) = \lim_{i \rightarrow +\infty} \mu_i(\{v_i\}).$$

Proof of Lemma 8.5.2. We have, by the very definition of μ_i :

$$\mu_i(\{v_i\}) = \lim_{j \rightarrow +\infty} p_{ij*}\gamma_{\overline{E}_j}(\{v_i\}).$$

For any integer $j \geq i$, the preimage $p_{ij}^{-1}(v_i)$ contains v_j . Therefore

$$(8.21) \quad p_{ij*}\gamma_{\overline{E}_j}(\{v_i\}) = \gamma_{\overline{E}_j}(p_{ij}^{-1}(v_i)) \geq \gamma_{\overline{E}_j}(v_j) = e^{-\pi\|v_j\|_{\overline{E}_j}^2}.$$

Besides, the inequality $q_{i*}\gamma_{\overline{E}_{i+1}} \leq e^{h_\theta^0(\overline{\ker q_i})} \gamma_{\overline{E}_i}$ established in Lemma 8.5.1 implies that:

$$p_{ij*}\gamma_{\overline{E}_j} \leq e^{\sum_{i \leq k < j} h_\theta^0(\overline{E}_k)} \gamma_{\overline{E}_i}.$$

Therefore,

$$(8.22) \quad p_{ij*}\gamma_{\overline{E}_j}(\{v_i\}) \leq e^{\sum_{i \leq k < j} h_\theta^0(\overline{E}_k)} \gamma_{\overline{E}_i}(\{v_i\}) = e^{-\pi\|v_i\|_{\overline{E}_i}^2 + \sum_{i \leq k < j} h_\theta^0(\overline{E}_k)}.$$

The estimates (8.20) follow from (8.21) and (8.22) by taking the limit $j \rightarrow +\infty$. \square

This completes the proof of part 1) of Theorem 8.3.4. The proof of part 2) will be based on the following:

Lemma 8.5.3. *For any η in \mathbb{R}_+^* ,*

$$(8.23) \quad \int_{\widehat{E}} f_\eta(x) d\mu_{\overline{E}_\bullet}(x) \geq \exp(M_{\overline{E}_\bullet}(1 + \eta)).$$

Proof. For any $(i, \eta) \in \mathbb{N} \times \mathbb{R}_+^*$, we have:

$$\begin{aligned} \int_{\widehat{E}} f_{i,\eta}(x) d\mu_{\overline{E}_\bullet}(x) &= \int_{\widehat{E}} e^{-\eta\pi\|p_i(x)\|_{\overline{E}_i}^2} d\mu_{\overline{E}_\bullet}(x) \\ &= \int_{E_i} e^{-\eta\pi\|v\|_{\overline{E}_i}^2} d\mu_i(v) \\ &= \lim_{j \rightarrow +\infty} \int_{E_i} e^{-\eta\pi\|v\|_{\overline{E}_i}^2} dp_{ij*} \gamma_{\overline{E}_j}(v) \\ &= \lim_{j \rightarrow +\infty} \int_{E_j} e^{-\eta\pi\|p_{ij}(w)\|_{\overline{E}_i}^2} d\gamma_{\overline{E}_j}(w). \end{aligned}$$

As the linear map $p_{ij,\mathbb{R}} : E_{j,\mathbb{R}} \rightarrow E_{i,\mathbb{R}}$ has an operator norm ≤ 1 with respect to the euclidean norms $\|\cdot\|_{\overline{E}_j}$ and $\|\cdot\|_{\overline{E}_i}$, we have:

$$\int_{E_j} e^{-\eta\pi\|p_{ij}(w)\|_{\overline{E}_i}^2} d\gamma_{\overline{E}_j}(w) \geq \int_{E_j} e^{-\eta\pi\|w\|_{\overline{E}_j}^2} d\gamma_{\overline{E}_j}(w) = \sum_{w \in E_j} e^{-\pi(1+\eta)\|w\|_{\overline{E}_j}^2}.$$

By taking the limit when $j \in \mathbb{N}_{\geq i}$ goes to $+\infty$, we obtain:

$$\int_{\widehat{E}} f_{i,\eta}(x) d\mu_{\overline{E}_\bullet}(x) \geq \exp(M_{\overline{E}_\bullet}(1 + \eta)).$$

Finally, we obtain (8.23) by taking the limit $i \rightarrow +\infty$, since by dominated convergence:

$$\lim_{i \rightarrow +\infty} \int_{\widehat{E}} f_{i,\eta}(x) d\mu_{\overline{E}_\bullet}(x) = \int_{\widehat{E}} f_\eta(x) d\mu_{\overline{E}_\bullet}(x).$$

□

To complete the proof of part 2) of Theorem 8.3.4, observe that, when η goes to 0, f_η converges pointwise towards $1_{E_{\mathbb{R}}^{\text{Hilb}}}$ (see (8.4)), and therefore, by dominated convergence:

$$\lim_{\eta \rightarrow 0+} \int_{\widehat{E}} f_\eta(x) d\mu_{\overline{E}_\bullet}(x) = \mu_{\overline{E}_\bullet}(\widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}).$$

If we now assume that $\lim_{t \rightarrow 1+} M_{\overline{E}_\bullet}(t) = M_{\overline{E}_\bullet}(1)$, we therefore deduce from (8.23):

$$\mu_{\overline{E}_\bullet}(\widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}) \geq \exp(M_{\overline{E}_\bullet}(1)).$$

The definition of $M_{\overline{E}_\bullet}(1)$, together with (8.8) and (8.9), show that

$$\exp(M_{\overline{E}_\bullet}(1)) = \mu_{\overline{E}_\bullet}(\widehat{E}),$$

and the previous inequality may also be written:

$$\mu_{\overline{E}_\bullet}(\widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}) \geq \mu_{\overline{E}_\bullet}(\widehat{E}).$$

This shows that

$$\mu_{\overline{E}_\bullet}(\widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}) = \mu_{\overline{E}_\bullet}(\widehat{E}),$$

or equivalently:

$$(8.24) \quad \mu_{\overline{E}_\bullet}(\widehat{E} \setminus \widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}) = 0.$$

Besides, as observed just before Proposition 7.2.5, the finiteness of $\bar{h}_\theta^0(\widehat{E})$, hence of \underline{h}_θ^0 implies that the set $\widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}$ is countable. Together with (8.24), this shows that

$$\mu_{\overline{E}_\bullet} = \sum_{v \in \widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}} \mu_{\overline{E}_\bullet}(\{v\}) \delta_v.$$

Combined with our previous computation (8.9) of $\mu_{\overline{E}_\bullet}(\{v\})$, this proves the equality (8.11):

$$\mu_{\overline{E}_\bullet} = \sum_{v \in \widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}} e^{-\pi \|v\|^2} \delta_v.$$

In particular

$$\mu_{\overline{E}_\bullet}(\widehat{E}) = \sum_{v \in \widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}} e^{-\pi \|v\|^2}.$$

By taking its logarithm, this relation becomes the equality (8.12): $\bar{h}_\theta^0(\widehat{E}) = \underline{h}_\theta^0(\widehat{E})$.

8.6. A converse theorem. In this section, we establish a converse to the results in the preceding paragraphs. Before formulating it, let us formulate some simple observations concerning the invariant $\bar{h}_\theta^0(\widehat{E})$ associated to some pro-hermitian vector bundle \widehat{E} .

Observe that, if \widehat{E} is the projective limit of some projective system \overline{E}_\bullet of surjective admissible morphism of hermitian vector bundles over $\text{Spec } \mathbb{Z}$, then, according to the very definition of $\bar{h}_\theta^0(\widehat{E})$, we have:

$$\bar{h}_\theta^0(\widehat{E}) \leq \liminf_{i \rightarrow +\infty} h_\theta^0(\overline{E}_i).$$

Moreover, this inequality is a sense optimal. Indeed there exists a decreasing sequence $(U_i)_{i \in \mathbb{N}}$ in $\mathcal{U}(\widehat{E})$, which constitutes a neighborhood basis of 0 in \widehat{E} , such that

$$\lim_{i \rightarrow +\infty} h_\theta^0(\overline{E}_{U_i}) = \bar{h}_\theta^0(\widehat{E}).$$

The pro-hermitian vector bundle $\widehat{E} \simeq \varprojlim_i \overline{E}_{U_i}$ may therefore be realised as the projective limit of some admissible projective system \overline{E}_\bullet such that

$$\lim_{i \rightarrow +\infty} h_\theta^0(\overline{E}_i) = \bar{h}_\theta^0(\widehat{E}).$$

Theorem 8.6.1. *Let $\widehat{E} := (\widehat{E}, E_{\mathbb{R}}^{\text{Hilb}}, \|\cdot\|)$ be a pro-hermitian vector bundle over $\text{Spec } \mathbb{Z}$ such that*

$$\underline{h}_\theta^0(\widehat{E}) = \bar{h}_\theta^0(\widehat{E}) < +\infty.$$

If \overline{E} is the projective limit of some admissible projective system \overline{E}_\bullet such that

$$\lim_{i \rightarrow +\infty} h_\theta^0(\overline{E}_i) = \bar{h}_\theta^0(\widehat{E}),$$

then there exist an increasing sequence of positive integers $i_\bullet := (i_k)_{k \in \mathbb{N}}$ such that the admissible projective system

$$\overline{E}_{i_\bullet} : \overline{E}_{i_0} \xleftarrow{p_{i_0 i_1}} \overline{E}_{i_1} \xleftarrow{p_{i_1 i_2}} \dots \xleftarrow{p_{i_{k-1} i_k}} \overline{E}_{i_k} \xleftarrow{p_{i_k i_{k+1}}} \overline{E}_{i_{k+1}} \xleftarrow{p_{i_{k+1} i_{k+2}}} \dots,$$

is summable.

As before, we have denoted by $p_{ij} : \overline{E}_i \rightarrow \overline{E}_j$ the admissible morphisms defining the projective system \overline{E}_\bullet .

Theorem 8.6.1 is actually a consequence of the following more precise result:

Lemma 8.6.2. *Under the assumptions of Theorem 8.6.1, for any $\epsilon \in \mathbb{R}_+^*$, there exists $(i(\epsilon), j(\epsilon)) \in \mathbb{N}^2$ for which the following condition is satisfied, for any $(i, j) \in \mathbb{N}^2$ such that $i \leq j$:*

$$i \geq i(\epsilon) \text{ and } j \geq j(\epsilon) \implies h_\theta^0(\overline{\ker p_{ij}}) < \epsilon.$$

Indeed, taking Lemma 8.6.2 for granted, for any sequence $(\eta_k)_{k \in \mathbb{N}}$ in $\mathbb{R}_+^{*\mathbb{N}}$ such that $\sum_{k \in \mathbb{N}} \eta_k < +\infty$, we may find an increasing sequence $(i_k)_{k \in \mathbb{N}}$ of positive integers such that, for any $k \in \mathbb{N}$, $i_k \geq i(\eta_k)$ and $i_{k+1} \geq j(\eta_k)$. Then, for any $k \in \mathbb{N}$, $h_\theta^0(\overline{\ker p_{i_k i_{k+1}}}) < \eta_k$, and the admissible projective system \overline{E}_{i_\bullet} is summable.

Proof of Lemma 8.6.2. We shall rely on Proposition C.2.3 of Appendix C, which we shall apply to the projective system of countable sets equipped with a finite measure:

$$(D_i, \gamma_i) := (E_i, \gamma_{\overline{E}_i}).$$

The sequence of total masses

$$\gamma_i(D_i) = \gamma_{\overline{E}_i}(E_i) = \exp h_\theta^0(\overline{E}_i)$$

is indeed convergent (of limit $\exp \widehat{h}_\theta^0(\widehat{E})$) in \mathbb{R}_+ and, inside $\widehat{D} = \widehat{E}$, we may consider the subset

$$\mathcal{C} := \widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}.$$

Since $\widehat{h}_\theta^0(\widehat{E}) < +\infty$, it is countable. For any $x \in \mathcal{C}$, we have

$$\gamma_j(p_j(x)) = e^{-\pi \|p_j(x)\|_{E_j}^2},$$

which converges to

$$\gamma(x) := e^{-\pi \|x\|^2}$$

when j goes to $+\infty$. Moreover

$$\sum_{x \in \mathcal{C}} \gamma(x) = \sum_{x \in \widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}} e^{-\pi \|x\|^2} = \exp \widehat{h}_\theta^0(\widehat{E})$$

is equal, by assumption, to

$$\exp \widehat{h}_\theta^0(\widehat{E}) = \lim_{i \rightarrow +\infty} \gamma_i(D_i).$$

This shows that the hypotheses of Proposition C.2.3 are fulfilled. Consequently, the sequence of measures $(\gamma_{\overline{E}_i})_{i \in \mathbb{N}}$ satisfies the condition **Conv**, and the associated limit measure on \widehat{E} is

$$\sum_{x \in \widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}} e^{-\pi \|x\|^2} \delta_x =: \mu_{\widehat{E}}.$$

As before, let us denote by $p_i : \widehat{E} \rightarrow E_i$ the canonical quotient map. The subset $\{0\}$ of \widehat{E} may be written as a countable decreasing intersection:

$$\{0\} = \bigcap_{i \in \mathbb{N}} p_i^{-1}(\{0\}),$$

and therefore

$$(8.25) \quad \mu_{\widehat{E}}(\{0\}) = \lim_{i \rightarrow +\infty} \mu_{\overline{E}_i}(p_i^{-1}(\{0\})).$$

Besides, since $\mu_{\widehat{E}}$ is the limit measure associated to $(\gamma_{\overline{E}_i})_{i \in \mathbb{N}}$, we have, for every $i \in \mathbb{N}$:

$$(8.26) \quad \begin{aligned} \mu_{\widehat{E}}(p_i^{-1}(\{0\})) &= p_{i*} \mu_{\widehat{E}}(\{0\}) = \lim_{j \in \mathbb{N}, j \rightarrow +\infty} p_{ij*} \gamma_{\overline{E}_j}(\{0\}) \\ &= \lim_{j \in \mathbb{N}, j \rightarrow +\infty} \gamma_{\overline{E}_j}(p_{ij}^{-1}(\{0\})) = \lim_{j \in \mathbb{N}, j \rightarrow +\infty} \exp h_\theta^0(\overline{\ker p_{ij}}). \end{aligned}$$

Since $\mu_{\widehat{E}}(\{0\}) = 1$, from (8.25) we derive the existence, for every $\epsilon \in \mathbb{R}_+^*$, of some $i(\epsilon) \in \mathbb{N}$ such that

$$\mu_{\widehat{E}}(p_{i(\epsilon)}^{-1}(0)) < e^\epsilon.$$

Then (8.26) shows the existence of an integer $j(\epsilon) \geq i(\epsilon)$ such that, for any $j \geq j(\epsilon)$,

$$h_\theta^0(\overline{\ker p_{i(\epsilon)j}}) < \epsilon.$$

The pair $(i(\epsilon), j(\epsilon))$ satisfies the conclusion of Lemma 8.6.2. Indeed, for any $(i, j) \in \mathbb{N}^2$ such that $i(\epsilon) \leq i \leq j$ and $j \geq j(\epsilon)$, we have $\ker p_{ij} \subset \ker p_{i(\epsilon)j}$ and therefore

$$h_\theta^0(\overline{\ker p_{ij}}) \leq h_\theta^0(\overline{\ker p_{i(\epsilon)j}}). \quad \square$$

8.7. Strongly summable and θ -finite pro-hermitian vector bundles. In this section, we formulate a few consequences of the main results in this section, Theorems 8.1.1 and 8.6.1.

These consequences, and the related definitions that follow them, are presented with a view toward Diophantine applications to be developed in the sequel of this article.

8.7.1. Strongly summable pro-hermitian vector bundles.

Corollary 8.7.1. *For any pro-hermitian vector bundle \widehat{E} over $\text{Spec } \mathcal{O}_K$, the following conditions **StS₁** and **StS₂** are equivalent:*

StS₁ : *There exists $\epsilon \in \mathbb{R}_+^*$ such that*

$$\underline{h}_\theta^0(\widehat{E} \otimes \overline{\mathcal{O}}(\epsilon)) = \overline{h}_\theta^0(\widehat{E} \otimes \overline{\mathcal{O}}(\epsilon)) < +\infty.$$

StS₂ : *There exists $\eta \in \mathbb{R}_+^*$ and a projective system of surjective admissible morphisms of hermitian vector bundles over $\text{Spec } \mathcal{O}_K$*

$$\overline{E}_\bullet : \overline{E}_0 \xleftarrow{q_0} \overline{E}_1 \xleftarrow{q_1} \dots \xleftarrow{q_{i-1}} \overline{E}_i \xleftarrow{q_i} \overline{E}_{i+1} \xleftarrow{q_{i+1}} \dots$$

such that $\widehat{E} \simeq \varprojlim_i \overline{E}_i$ and such that the projective system $\overline{E}_\bullet \otimes \overline{\mathcal{O}}(\eta)$ is summable.

Proof. Theorem 8.1.1 shows that, when Condition (ii) holds for some $\eta \in \mathbb{R}_+^*$, then Condition (i) holds for any $\epsilon \in]0, \eta[$.

Conversely, assume that Condition (i) is satisfied for some $\epsilon \in \mathbb{R}_+^*$. As already observed, by the very definition of $\underline{h}_\theta^0(\widehat{E} \otimes \overline{\mathcal{O}}(\epsilon))$, there exists a projective system \overline{F}_\bullet of surjective admissible morphisms of hermitian vector bundles over $\text{Spec } \mathcal{O}_K$ such that $\widehat{E} \otimes \overline{\mathcal{O}}(\epsilon) \simeq \varprojlim_i \overline{F}_i$ and

$$\lim_{i \rightarrow +\infty} \underline{h}_\theta^0(\overline{F}_i) = \underline{h}_\theta^0(\widehat{E} \otimes \overline{\mathcal{O}}(\epsilon)).$$

Theorem 8.6.1, applied to the pro-hermitian vector bundle $\pi_* \widehat{E} \otimes \overline{\mathcal{O}}(\epsilon)$ and to the projective system $\pi_* \overline{F}_\bullet$ over $\text{Spec } \mathbb{Z}$, shows the existence of some increasing sequence of positive integers i_\bullet such that the projective system $\pi_* \overline{F}_{i_\bullet}$ — or equivalently \overline{F}_{i_\bullet} — is summable.

Finally Condition (ii) is satisfied by $\eta := \epsilon$ and by $\overline{E}_\bullet := \overline{F}_{i_\bullet} \otimes \overline{\mathcal{O}}(-\epsilon)$. \square

When the conditions **StS₁** and **StS₂** in Corollary 8.7.1 are satisfied, we shall say that the pro-hermitian vector bundle \widehat{E} is *strongly summable*. According to **StS₂**, the strongly summable pro-hermitian vector bundles are precisely those that can be realized as projective limits of the strongly summable projective systems of surjective admissible morphisms of hermitian vector bundles defined in paragraph 8.2.2.

Clearly, if \widehat{E} is strongly summable, then, for any δ in \mathbb{R}_+ , $\widehat{E} \otimes \overline{\mathcal{O}}(-\delta)$ is strongly summable and

$$\underline{h}_\theta^0(\widehat{E} \otimes \overline{\mathcal{O}}(-\delta)) = \overline{h}_\theta^0(\widehat{E} \otimes \overline{\mathcal{O}}(-\delta)) < +\infty.$$

Then we shall write $h_\theta^0(\widehat{E})$ instead of $\underline{h}_\theta^0(\widehat{E})$ or $\overline{h}_\theta^0(\widehat{E})$.

8.7.2. *θ -finite pro-hermitian and Hilbertisable vector bundles.*

Corollary 8.7.2. *Let $\widehat{E} := (\widehat{E}, (E_\sigma^{\text{Hilb}}, \|\cdot\|_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}})$ be a pro-hermitian vector bundle over $\text{Spec } \mathcal{O}_K$, and let $\tilde{E} := (\widehat{E}, (E_\sigma^{\text{Hilb}})_{\sigma:K \hookrightarrow \mathbb{C}})$ be the associated object in $\text{pro}\widetilde{\text{Vect}}(\mathcal{O}_K)$.*

The following conditions are equivalent:

(i) *For any pro-hermitian vector bundle $\widehat{E}' := (\widehat{E}, (E_\sigma^{\text{Hilb}}, \|\cdot\|'_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}})$ admitting \tilde{E} as associated object in $\text{pro}\widetilde{\text{Vect}}(\mathcal{O}_K)$,*

$$(8.27) \quad \underline{h}_\theta^0(\widehat{E}') = \overline{h}_\theta^0(\widehat{E}') < +\infty.$$

(ii) *For any $\delta \in \mathbb{R}$,*

$$\underline{h}_\theta^0(\widehat{E} \otimes \overline{\mathcal{O}}(\delta)) = \overline{h}_\theta^0(\widehat{E} \otimes \overline{\mathcal{O}}(\delta)) < +\infty.$$

Proof. For any $\delta \in \mathbb{R}$, the pro-hermitian vector bundles \widehat{E} and $\widehat{E} \otimes \overline{\mathcal{O}}(\delta)$ define the same object in $\text{pro}\widetilde{\text{Vect}}(\mathcal{O}_K)$. Therefore (i) implies (ii).

Conversely, let us assume that (ii) holds and consider a pro-hermitian vector bundle $\widehat{E}' := (\widehat{E}, (E_\sigma^{\text{Hilb}}, \|\cdot\|'_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}})$ admitting \tilde{E} as associated object in $\text{pro}\widetilde{\text{Vect}}(\mathcal{O}_K)$. There exists $\lambda \in \mathbb{R}_+^*$ such that the identity maps on \widehat{E} and on the E_σ^{Hilb} 's define a morphism in $\text{Hom}_{\mathcal{O}_K}^{\leq \lambda}(\widehat{E}', \widehat{E})$, or equivalently in $\text{Hom}_{\mathcal{O}_K}^{\leq 1}(\widehat{E}', \widehat{E} \otimes \overline{\mathcal{O}}(\log \lambda))$. According to (ii), $\widehat{E} \otimes \overline{\mathcal{O}}(\log \lambda)$ satisfies **StS**₁, and therefore, by Corollary 8.7.1, it satisfies **StS**₂. This implies that \widehat{E}' also satisfies **StS**₂, and is therefore strongly summable, and consequently that (8.27) holds. \square

When the equivalent conditions in Corollary 8.7.2 are realized — that is, when $\widehat{E} \otimes \overline{\mathcal{O}}(\delta)$ is strongly summable for every $\delta \in \mathbb{R}$ — we shall say that \widehat{E} and \tilde{E} are *θ -finite*.

This terminology makes reference to the fact that, to any θ -finite \widehat{E} as above, one may associate its theta function

$$\theta_{\widehat{E}}: \mathbb{R}_+^* \longrightarrow [1, +\infty[$$

defined, for any $t \in \mathbb{R}_+^*$, by:

$$\theta_{\widehat{E}}(t) := \exp \left(h_\theta^0(\widehat{E} \otimes \overline{\mathcal{O}}(-(\log t)/2)) \right).$$

According to (8.12), we have:

$$\theta_{\widehat{E}}(t) = \sum_{v \in \widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}} e^{-\pi t \|v\|^2},$$

where as usual, $\|\cdot\|$ denotes the Hilbert norm on $E_{\mathbb{C}}^{\text{Hilb}} \simeq \bigoplus_{\sigma:K \hookrightarrow \mathbb{C}} E_\sigma^{\text{Hilb}}$ defined by

$$\|(v_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}}\|^2 := \sum_{\sigma:K \hookrightarrow \mathbb{C}} \|v_\sigma\|_\sigma^2.$$

Observe that, for any θ -finite pro-hermitian vector bundle \widehat{E} , the subgroup $\widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}$ is discrete in the Hilbert space $(E_{\mathbb{R}}^{\text{Hilb}}, \|\cdot\|)$. (This directly follows from Proposition 7.2.5.) Consequently, as discussed in Section 6.8 (see notably Corollary 6.8.3), it defines an *ind*-euclidean lattice:

$$\widehat{E}^{\text{ind}} := (\widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}, \|\cdot\|).$$

This ind-euclidean lattice is easily seen to satisfy the equivalent conditions in Proposition 7.1.2. Actually, for any $\delta \in \mathbb{R}$, the following equality holds:

$$h_\theta^0(\widehat{E}^{\text{ind}} \otimes \overline{\mathcal{O}}(\delta)) = h_\theta^0(\widehat{E} \otimes \overline{\mathcal{O}}(\delta)) = \log \theta_{\widehat{E}}(e^{-2\delta}).$$

8.7.3. *Examples.* From the computations in paragraphs 7.4.1 and 7.4.2, we immediately obtain explicit examples of θ -summable pro-hermitian vector bundles:

Proposition 8.7.3. 1) *With the notation of paragraph 7.4.1, for any element $\lambda := (\lambda_i)_{i \in \mathbb{N}}$ of $\mathbb{R}_+^{*\mathbb{N}}$, the pro-euclidean lattice \widehat{V}_λ is θ -summable if and only if, for every $t \in \mathbb{R}_+^*$,*

$$\sum_{i \in \mathbb{N}} e^{-t\lambda_i} < +\infty.$$

2) *For any positive real number R , the pro-euclidean lattice $\widehat{H}(R)$ is θ -summable if and only if $R > 1$.* \square

8.7.4. *A permanence property.* We conclude this section by showing that the property, for a pro-hermitian vector bundle, of being strongly summable or θ -finite is inherited by its “closed sub-bundles”.

Proposition 8.7.4. *Let $f : \widehat{E} \rightarrow \widehat{F}$ be a morphism of pro-hermitian vector bundles over $\text{Spec } \mathcal{O}_K$ such that the underlying morphism of topological \mathcal{O}_K -modules $\hat{f} : \widehat{E} \rightarrow \widehat{F}$ is injective and strict.*

- 1) *If f belongs to $\text{Hom}_{\mathcal{O}_K}^{\leq 1}(\widehat{E}, \widehat{F})$ and if \widehat{F} is strongly summable, then \widehat{E} is strongly summable.*
- 2) *If \widehat{F} is θ -finite, then \widehat{E} is θ -finite.*

Proof. 1) Let us assume that \widehat{F} is strongly summable, and hence satisfies **StS₂**. Then we may choose a defining sequence $(V_i)_{i \in \mathbb{N}}$ in $\mathcal{U}(\widehat{F})$ and $\epsilon > 0$ such that, if we let $\overline{F}_i := \overline{F}_{V_i}$, the admissible surjective morphisms

$$r_i := p_{V_i V_{i+1}} : \overline{F}_{i+1} \rightarrow \overline{F}_i$$

satisfy

$$\sum_{i \in \mathbb{N}} h_\theta^0(\ker r_i \otimes \overline{\mathcal{O}}(\epsilon)) < +\infty.$$

We may apply Proposition 5.4.12, part 1), to the strict injective morphism $\hat{f} : \widehat{E} \rightarrow \widehat{F}$ in $CTC_{\mathcal{O}_K}$ and to the defining sequence $(V_i)_{i \in \mathbb{N}}$. Therefore the sequence $(U_i)_{i \in \mathbb{N}} := (\hat{f}^{-1}(V_i))_{i \in \mathbb{N}}$ is a defining sequence in $\mathcal{U}(\widehat{E})$, and we may consider the hermitian vector bundles $\overline{E}_i := \overline{E}_{U_i}$ and the injective morphisms of finitely generated projective \mathcal{O}_K -modules

$$f_i := E_i := \widehat{E}/U_i \rightarrow F_i := \widehat{F}/V_i$$

induced by \hat{f} .

Let us also assume that f belongs to $\text{Hom}_{\mathcal{O}_K}^{\leq 1}(\widehat{E}, \widehat{F})$. Then, for every $i \in \mathbb{N}$, f_i belongs to $\text{Hom}_{\mathcal{O}_K}^{\leq 1}(\overline{E}_i, \overline{F}_i)$. Indeed the hermitian norms defining \overline{E}_i and \overline{F}_i are the quotient norms of the Hilbert space norms defining \widehat{E} and \widehat{F} .

Let us consider the surjective admissible morphisms of hermitian vector bundles over $\text{Spec } \mathcal{O}_K$:

$$q_i := p_{U_i U_{i+1}} : \overline{E}_{i+1} \rightarrow \overline{E}_i.$$

For any $i \in \mathbb{N}$, the commutativity of the diagram

$$(8.28) \quad \begin{array}{ccc} \overline{E}_{i+1} & \xrightarrow{q_i} & \overline{E}_i \\ f_{i+1} \downarrow & & \downarrow f_i \\ \overline{F}_{i+1} & \xrightarrow{r_i} & \overline{F}_i \end{array}$$

shows that f_{i+1} defines a morphism

$$f_{i+1} : \ker q_i \longrightarrow \ker r_i$$

which is injective and belongs to $\text{Hom}_{\mathcal{O}_K}^{\leq 1}(\overline{\ker q_i}, \overline{\ker r_i})$.

Therefore

$$h_\theta^0(\overline{\ker q_i} \otimes \overline{\mathcal{O}(\epsilon)}) \leq h_\theta^0(\overline{\ker r_i} \otimes \overline{\mathcal{O}(\epsilon)})$$

(by Proposition 3.3.2, 1)) and finally, we obtain:

$$\sum_{i \in \mathbb{N}} h_\theta^0(\overline{\ker q_i} \otimes \overline{\mathcal{O}(\epsilon)}) \leq \sum_{i \in \mathbb{N}} h_\theta^0(\overline{\ker r_i} \otimes \overline{\mathcal{O}(\epsilon)}) < +\infty.$$

Thus \widehat{E} also satisfies **StS₂**.

2) For any embedding $\sigma : K \hookrightarrow \mathbb{C}$, let us denote by $\|f_\sigma\|$ the operator norm of the continuous linear map f_σ between the Hilbert spaces $(E_\sigma^{\text{Hilb}}, \|\cdot\|_{\widehat{E}, \sigma})$ and $(F_\sigma^{\text{Hilb}}, \|\cdot\|_{\widehat{F}, \sigma})$, and let us choose a real number such that

$$\lambda \geq \max_{\sigma: K \hookrightarrow \mathbb{C}} \log \|f_\sigma\|.$$

Then f belongs to $\text{Hom}_{\mathcal{O}_K}^{\leq 1}(\widehat{E}, \widehat{F} \otimes \overline{\mathcal{O}(\lambda)})$ and therefore, for any $\delta \in \mathbb{R}$, to $\text{Hom}_{\mathcal{O}_K}^{\leq 1}(\widehat{E} \otimes \overline{\mathcal{O}(\delta)}, \widehat{F} \otimes \overline{\mathcal{O}(\lambda + \delta)})$.

Let us assume that \widehat{F} is θ -finite. Then, for any $\delta \in \mathbb{R}$, $\widehat{F} \otimes \overline{\mathcal{O}(\lambda + \delta)}$ is strongly summable, and therefore, according to Part 1), $\widehat{E} \otimes \overline{\mathcal{O}(\delta)}$ is strongly summable. This proves that \widehat{E} is θ -finite. \square

9. SUBADDITIVITY PROPERTIES OF THE θ -INVARIANTS ATTACHED TO INFINITE DIMENSIONAL HERMITIAN VECTOR BUNDLES

In this final section, we introduce the notions of *short exact sequences* and *admissible short exact sequences* of pro-hermitian vector bundles, and we extend the subadditivity properties of the θ -invariants, previously considered in section 3.8 and paragraph 4.3.1 for hermitian vector bundles, to the infinite dimensional setting.

We also give some applications of these subadditivity properties to strongly summable and θ -finite pro-hermitian vector bundles.

We still denote by K a number field, by \mathcal{O}_K its ring of integers, and by $\pi : \text{Spec } \mathcal{O}_K \longrightarrow \text{Spec } \mathbb{Z}$ the morphism of schemes from $\text{Spec } \mathcal{O}_K$ to $\text{Spec } \mathbb{Z}$.

9.1. Short exact sequences of infinite dimensional hermitian vector bundles.

9.1.1. *Short exact sequences of pro-hermitian vector bundles.* We define a *short exact sequence of pro-hermitian vector bundles over $\text{Spec } \mathcal{O}_K$* as a diagram

$$(9.1) \quad 0 \longrightarrow \widehat{E} \xrightarrow{i} \widehat{F} \xrightarrow{q} \widehat{G} \longrightarrow 0$$

in $\text{proVect}^{\text{cont}}(\mathcal{O}_K)$ such that:

(1) the diagram of \mathcal{O}_K -modules

$$(9.2) \quad 0 \longrightarrow \widehat{E} \xrightarrow{\hat{i}} \widehat{F} \xrightarrow{\hat{q}} \widehat{G} \longrightarrow 0$$

is a short exact sequence, and

(2) for every embedding $\sigma : K \hookrightarrow \mathbb{C}$, the complex of \mathbb{C} -vector spaces

$$(9.3) \quad 0 \longrightarrow E_\sigma^{\text{Hilb}} \xrightarrow{i_\sigma} F_\sigma^{\text{Hilb}} \xrightarrow{q_\sigma} G_\sigma^{\text{Hilb}} \longrightarrow 0.$$

is a short exact sequence.

Let us formulate a few observations concerning this definition:

(i) It is straightforward that the diagram (9.1) in $\text{pro}\overline{\text{Vect}}^{\text{cont}}(\mathcal{O}_K)$ is a short exact sequence of pro-hermitian vector bundles over $\text{Spec } \mathcal{O}_K$ if and only if the diagram

$$0 \longrightarrow \pi_* \widehat{E} \xrightarrow{i} \pi_* \widehat{F} \xrightarrow{q} \pi_* \widehat{G} \longrightarrow 0$$

is a short exact sequence of pro-hermitian vector bundles over $\text{Spec } \mathbb{Z}$.

(ii) The exactness of the diagram (9.2) implies it is actually a *split* short exact sequence of topological \mathcal{O}_K -module. Namely there exists a continuous morphism of topological \mathcal{O}_K -modules

$$\hat{s} : \widehat{G} \longrightarrow \widehat{F}$$

such that

$$\hat{q} \circ \hat{s} = \text{Id}_{\widehat{G}}.$$

This follows from the results on strict morphisms in CTC_A established in Section 5.4, applied to the ring $A = \mathcal{O}_K$: this ring satisfies **Ded**₃ and therefore, according to Proposition 5.4.10, the morphisms \hat{i} and \hat{q} are strict; therefore, by Proposition 5.4.5, the short exact sequence (9.2) is split.

In particular, the diagram (9.2) remains exact — actually split — after any “completed base change”. Notably, for every embedding $\sigma : K \hookrightarrow \mathbb{C}$, the diagram

$$0 \longrightarrow \widehat{E}_\sigma \xrightarrow{\hat{i}_\sigma} \widehat{F}_\sigma \xrightarrow{\hat{q}_\sigma} \widehat{G}_\sigma \longrightarrow 0$$

is a split short exact sequence of complex Fréchet spaces.

(iii) Similarly, according to Banach’s open mapping theorem and to basic results on Hilbert spaces, the morphism i_σ and p_σ in the diagram (9.3) are strict, and this diagram is actually split in the category of topological vector spaces.

We shall say that the short exact sequence (9.1) in $\text{pro}\overline{\text{Vect}}^{\text{cont}}(\mathcal{O}_K)$ is *admissible* when the maps $i_\sigma : E_\sigma^{\text{Hilb}} \longrightarrow F_\sigma^{\text{Hilb}}$ (resp. $q_\sigma : F_\sigma^{\text{Hilb}} \longrightarrow G_\sigma^{\text{Hilb}}$) are isometries (resp. co-isometries³⁰) from $(E_\sigma^{\text{Hilb}}, \|\cdot\|_\sigma^{\text{Hilb}})$ to $(F_\sigma^{\text{Hilb}}, \|\cdot\|_\sigma^{\text{Hilb}})$ (resp. from $(F_\sigma^{\text{Hilb}}, \|\cdot\|_\sigma^{\text{Hilb}})$ to $(G_\sigma^{\text{Hilb}}, \|\cdot\|_\sigma^{\text{Hilb}})$).

9.1.2. Short exact sequences of ind-hermitian vector bundles. Duality. We define a *short exact sequence of ind-hermitian vector bundles over $\text{Spec } \mathcal{O}_K$* as a diagram

$$(9.4) \quad 0 \longrightarrow \overline{E} \xrightarrow{i} \overline{F} \xrightarrow{q} \overline{G} \longrightarrow 0$$

in $\text{ind}\overline{\text{Vect}}^{\text{cont}}(\mathcal{O}_K)$ such that:

(i) the diagram of \mathcal{O}_K -modules

$$(9.5) \quad 0 \longrightarrow E \xrightarrow{i} F \xrightarrow{q} G \longrightarrow 0$$

is a short exact sequence, and

(ii) for every embedding $\sigma : K \hookrightarrow \mathbb{C}$, the diagram of \mathbb{C} -vector spaces

$$(9.6) \quad 0 \longrightarrow E_\sigma \xrightarrow{i_\sigma} F_\sigma \xrightarrow{q_\sigma} G_\sigma \longrightarrow 0$$

³⁰In other words, their adjoints q_σ^* are isometries.

is a short exact sequence, and i_σ and q_σ are strict morphisms of topological vector spaces when E_σ , F_σ , and G_σ are equipped with the topology defined by the hermitian norms $\|\cdot\|_{E,\sigma}$, $\|\cdot\|_{F,\sigma}$, and $\|\cdot\|_{G,\sigma}$ that define $\overline{E} := (E, (\|\cdot\|_{E,\sigma})_{\sigma:K \hookrightarrow \mathbb{C}})$, $\overline{F} := (F, (\|\cdot\|_{F,\sigma})_{\sigma:K \hookrightarrow \mathbb{C}})$ and $\overline{G} := (G, (\|\cdot\|_{G,\sigma})_{\sigma:K \hookrightarrow \mathbb{C}})$.

The last conditions means that the norms $\|\cdot\|_{E,\sigma}$ and $\|i_\sigma(\cdot)\|_{F,\sigma}$ on E_σ are equivalent, and that the norm $\|\cdot\|_{G,\sigma}$ and the norm quotient of $\|\cdot\|_{F,\sigma}$ via q_σ on G_σ are equivalent.

We shall say that the short exact sequence (9.4) is *admissible* when the norms $\|\cdot\|_{E,\sigma}$ and $\|i_\sigma(\cdot)\|_{F,\sigma}$ on E_σ coincide, and also the norm $\|\cdot\|_{G,\sigma}$ and the norm quotient of $\|\cdot\|_{F,\sigma}$ on G_σ .

These definitions are compatible, via the duality between $\text{ind}\overline{\text{Vect}}^{\text{cont}}(\mathcal{O}_K)$ and $\text{pro}\overline{\text{Vect}}^{\text{cont}}(\mathcal{O}_K)$, with the definitions of short exact sequences in $\text{pro}\overline{\text{Vect}}^{\text{cont}}(\mathcal{O}_K)$ in the previous paragraph.

Indeed, by combining the basic facts concerning the duality of ind- and pro-hermitian vector bundles presented in Section 6.5, the results on exact sequences in CP_A and CTC_A established paragraph 5.4.3 (in the special case where $A = \mathcal{O}_K$, and therefore satisfies **Ded**₃), and the basic theory of Hilbert spaces, one may establish the following proposition, the proof of which is left to the reader:

Proposition 9.1.1. *Let \overline{E} , \overline{F} and \overline{G} be three ind-hermitian vector bundles over $\text{Spec } \mathcal{O}_K$, and let \overline{E}^\vee , \overline{F}^\vee and \overline{G}^\vee be the dual pro-hermitian vector bundles over $\text{Spec } \mathcal{O}_K$.*

For any two morphisms $i : \overline{E} \rightarrow \overline{F}$ and $q : \overline{F} \rightarrow \overline{G}$ of ind-hermitian vector bundles over $\text{Spec } \mathcal{O}_K$, the diagram

$$0 \longrightarrow \overline{E} \xrightarrow{i} \overline{F} \xrightarrow{q} \overline{G} \longrightarrow 0$$

is a short exact sequence (resp. an admissible short exact sequence) of ind-hermitian vector bundles over $\text{Spec } \mathcal{O}_K$ if and only if the dual diagram

$$0 \longrightarrow \overline{G} \xrightarrow{q^\vee} \overline{F}^\vee \xrightarrow{i^\vee} \overline{E}^\vee \longrightarrow 0$$

is a short exact sequence (resp. an admissible short exact sequence) of pro-hermitian vector bundles over $\text{Spec } \mathcal{O}_K$. \square

9.2. Short exact sequences and θ -invariants of pro-hermitian vector bundles.

9.2.1. This section is devoted to the following extension to infinite dimensional hermitian vector bundles of the sub-additivity of the θ -invariants established for finite rank hermitian vector bundles in Section 3.8 and Paragraph 4.3.1.

Theorem 9.2.1. *Consider an admissible short exact sequence of pro-hermitian vector bundles over the arithmetic curve $\text{Spec } \mathcal{O}_K$:*

$$0 \longrightarrow \widehat{E} \xrightarrow{i} \widehat{F} \xrightarrow{q} \widehat{G} \longrightarrow 0$$

1) *The following inequalities hold in $[0, +\infty]$:*

$$(9.7) \quad \underline{h}_\theta^0(\widehat{F}) \leq \underline{h}_\theta^0(\widehat{E}) + \underline{h}_\theta^0(\widehat{G}),$$

$$(9.8) \quad \overline{h}_\theta^0(\widehat{F}) \leq \overline{h}_\theta^0(\widehat{E}) + \overline{h}_\theta^0(\widehat{G}),$$

and

$$(9.9) \quad h_\theta^1(\widehat{F}) \leq h_\theta^1(\widehat{E}) + h_\theta^1(\widehat{G}),$$

2) *If \widehat{E} has finite rank (or equivalently, is defined by some hermitian vector bundle \overline{E} over $\text{Spec } \mathcal{O}_K$), then we also have:*

$$(9.10) \quad \underline{h}_\theta^0(\widehat{G}) \leq \underline{h}_\theta^0(\widehat{F}) - \widehat{\deg} \pi_* \overline{E}.$$

3) If \widehat{G} has finite rank (or equivalently, is defined by some hermitian vector bundle \overline{G} over $\text{Spec } \mathcal{O}_K$), then

$$(9.11) \quad h_\theta^1(\widehat{E}) \leq h_\theta^1(\widehat{F}) + \widehat{\deg} \pi_* \overline{G}.$$

By means of arguments similar to the one in the proof of part 1) of Theorem 9.3.1 *infra*, one may also show that, in the situation of part 2) of Theorem 9.2.1 (that is when \widehat{E} is some hermitian vector bundle \overline{E}), we also have:

$$(9.12) \quad \overline{h}_\theta^0(\widehat{G}) \leq \lim_{\epsilon \rightarrow 0_+} \overline{h}_\theta^0(\widehat{F} \otimes \overline{\mathcal{O}}(\epsilon)) - \widehat{\deg} \pi_* \overline{E}.$$

We leave this to the interested reader.

For later reference, let us record some straightforward consequences of Theorem 9.2.1 and of the estimates previously established in Proposition 7.3.5.

Corollary 9.2.2. *Let us keep the notation of Theorem 9.2.1.*

1) Let us assume that \widehat{E} has finite rank. Then $\underline{h}_\theta^0(\widehat{F}) < +\infty$ if and only if $\underline{h}_\theta^0(\widehat{G}) < +\infty$.

2) Let us assume that \widehat{G} has finite rank. Then $\underline{h}_\theta^0(\widehat{E}) < +\infty$ if and only if $\underline{h}_\theta^0(\widehat{F}) < +\infty$, and $\overline{h}_\theta^0(\widehat{E}) < +\infty$ if and only if $\overline{h}_\theta^0(\widehat{F}) < +\infty$. \square

9.2.2. *Proof of Theorem 9.2.1: I. Preliminary.* Let us consider an admissible short exact sequence of pro-hermitian vector bundles over the arithmetic curve $\text{Spec } \mathcal{O}_K$:

$$(9.13) \quad 0 \longrightarrow \widehat{E} \xrightarrow{i} \widehat{F} \xrightarrow{q} \widehat{G} \longrightarrow 0$$

By the very definition of a short exact sequence in $\text{pro}\overline{\text{Vect}}^{\text{cont}}(\mathcal{O}_K)$, from (9.13), we derive a commutative diagram with exact lines, where the vertical arrows denote the inclusion maps defining the pro-hermitian vector bundles \widehat{E} , \widehat{F} , and \widehat{G} :

$$(9.14) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{E} & \xrightarrow{\hat{i}} & \widehat{F} & \xrightarrow{\hat{q}} & \widehat{G} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widehat{E}_{\mathbb{R}} & \xrightarrow{\hat{i}_{\mathbb{R}}} & \widehat{F}_{\mathbb{R}} & \xrightarrow{\hat{q}_{\mathbb{R}}} & \widehat{G}_{\mathbb{R}} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & E_{\mathbb{R}}^{\text{Hilb}} & \xrightarrow{i_{\mathbb{R}}} & F_{\mathbb{R}}^{\text{Hilb}} & \xrightarrow{q_{\mathbb{R}}} & G_{\mathbb{R}}^{\text{Hilb}} & \longrightarrow & 0 \end{array}$$

Lemma 9.2.3. *The commutative diagrams*

$$(9.15) \quad \begin{array}{ccc} \widehat{E} & \xrightarrow{\hat{i}} & \widehat{F} \\ \downarrow & & \downarrow \\ \widehat{E}_{\mathbb{R}} & \xrightarrow{\hat{i}_{\mathbb{R}}} & \widehat{F}_{\mathbb{R}} \end{array}$$

and

$$(9.16) \quad \begin{array}{ccc} \widehat{E}_{\mathbb{R}} & \xrightarrow{\hat{i}_{\mathbb{R}}} & \widehat{F}_{\mathbb{R}} \\ \uparrow & & \uparrow \\ E_{\mathbb{R}}^{\text{Hilb}} & \xrightarrow{i_{\mathbb{R}}} & F_{\mathbb{R}}^{\text{Hilb}} \end{array}$$

extracted from (9.14) are cartesian squares³¹.

³¹of sets, and also respectively of topological \mathbb{Z} -modules and of topological \mathbb{R} -vector spaces.

Proof. The cartesian character of (9.15) (resp., of (9.16)) follows from the exactness of the first (resp., of the last) two lines in the commutative diagram (9.14). \square

As already observed in 9.1.1, we may choose a continuous \mathcal{O}_K -linear splitting

$$\hat{s} : \hat{G} \longrightarrow \hat{F}$$

of the map $\hat{p} : \hat{F} \longrightarrow \hat{E}$.

For any $U \in \mathcal{U}(\hat{E})$ and any $W \in \mathcal{U}(\hat{G})$, the (direct) sum

$$V := \hat{i}(U) + \hat{s}(W)$$

belongs to $\mathcal{U}(\hat{F})$. The quotients $E_U := \hat{E}/U$, $F_V := \hat{F}/V$ and $G_W := \hat{G}/W$ fit into a commutative diagram of \mathcal{O}_K -modules

$$(9.17) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \hat{E} & \xrightarrow{\hat{i}} & \hat{F} & \xrightarrow{\hat{q}} & \hat{G} \longrightarrow 0 \\ & & p_U \downarrow & & p_V \downarrow & & p_W \downarrow \\ 0 & \longrightarrow & E_U & \xrightarrow{i_{UW}} & F_V & \xrightarrow{q_{UW}} & G_W \longrightarrow 0, \end{array}$$

where the vertical arrows denote the canonical surjections.

Besides, E_U , F_V and G_W are the finitely generated projective \mathcal{O}_K -modules underlying the hermitian vector bundles \overline{E}_U , \overline{F}_V and \overline{G}_W defined from the pro-hermitian vector bundles \hat{E} , \hat{F} and \hat{G} . The real vector spaces $E_{U,\mathbb{R}}$, $F_{V,\mathbb{R}}$ and $G_{W,\mathbb{R}}$ may be endowed with the euclidean norms $\|\cdot\|_{\pi_*\overline{E}_U}$, $\|\cdot\|_{\pi_*\overline{F}_V}$ and $\|\cdot\|_{\pi_*\overline{G}_W}$.

The norm $\|\cdot\|_{\pi_*\overline{E}_U}$ is characterized by the fact that the composite map

$$p'_{U,\mathbb{R}} : E_{\mathbb{R}}^{\text{Hilb}} \hookrightarrow \hat{E}_{\mathbb{R}} \xrightarrow{p_{U,\mathbb{R}}} E_{U,\mathbb{R}}$$

— which is onto since the image of $E_{\mathbb{R}}^{\text{Hilb}}$ is dense in $\hat{E}_{\mathbb{R}}$ — is a co-isometry when $E_{\mathbb{R}}^{\text{Hilb}}$ is equipped with the Hilbert norm attached to \hat{E} and $E_{U,\mathbb{R}}$ is equipped with $\|\cdot\|_{\pi_*\overline{E}_U}$. Similar remarks apply to $\|\cdot\|_{\pi_*\overline{F}_V}$ and $\|\cdot\|_{\pi_*\overline{G}_W}$. Finally, from the last two lines of (9.14), we deduce a commutative diagram

$$(9.18) \quad \begin{array}{ccccccc} 0 & \longrightarrow & E_{U,\mathbb{R}} & \xrightarrow{i_{UW,\mathbb{R}}} & F_{V,\mathbb{R}} & \xrightarrow{q_{UW,\mathbb{R}}} & G_{W,\mathbb{R}} \longrightarrow 0 \\ & & p'_{U,\mathbb{R}} \uparrow & & p'_{V,\mathbb{R}} \uparrow & & p'_{W,\mathbb{R}} \uparrow \\ 0 & \longrightarrow & E_{\mathbb{R}}^{\text{Hilb}} & \xrightarrow{i_{\mathbb{R}}} & F_{\mathbb{R}}^{\text{Hilb}} & \xrightarrow{q_{\mathbb{R}}} & G_{\mathbb{R}}^{\text{Hilb}} \longrightarrow 0 \end{array}$$

where the vertical arrows are co-isometries.

Lemma 9.2.4. *Let us keep the previous notation.*

1) *For any $e \in E_{U,\mathbb{R}}$, we have*

$$(9.19) \quad \|i_{UW,\mathbb{R}}(e)\|_{\pi_*\overline{F}_V} \leq \|e\|_{\pi_*\overline{E}_U}.$$

Moreover $\|i_{UW,\mathbb{R}}(e)\|_{\pi_\overline{F}_V}$ is a non-increasing function of $W \in \mathcal{U}(\hat{G})$ and*

$$(9.20) \quad \lim_{W \in \mathcal{U}(\hat{G})} \|i_{UW,\mathbb{R}}(e)\|_{\pi_*\overline{F}_V} = \|e\|_{\pi_*\overline{E}_U}.$$

2) *The map $q_{UW,\mathbb{R}} : F_{V,\mathbb{R}} \longrightarrow G_{W,\mathbb{R}}$ is a co-isometry.*

The proof of (9.20) will rely on the following simple proposition concerning orthogonal projections in Hilbert spaces, that we shall leave as an exercise:

Proposition 9.2.5. *Let $(H, \|\cdot\|)$ be a (real or complex) Hilbert space and let*

$$H_0 \supseteq H_1 \supseteq H_2 \supseteq H_3 \supseteq \dots \supseteq H_n \supseteq H_{n+1} \dots$$

be a decreasing sequence of closed vector subspaces of H .

For any $n \in \mathbb{N}$, let

$$p_n : H \longrightarrow H_n^\perp$$

denote the orthogonal projection from H onto the orthogonal complement H_n^\perp of H_n in H , and let

$$p_\infty : H \longrightarrow \left(\bigcap_{n \in \mathbb{N}} H_n \right)^\perp$$

be the orthogonal projection onto the orthogonal complement of $\bigcap_{n \in \mathbb{N}} H_n$.

Then, for any $x \in H$,

$$\lim_{n \rightarrow +\infty} \|p_n(x) - p_\infty(x)\| = 0,$$

and $(\|p_n(x)\|)_{n \in \mathbb{N}}$ is a non-decreasing sequence of limit $\|p_\infty(x)\|$. □

Proof of Lemma 9.2.4. 1) The estimate (9.19) follows from the fact that, in the commutative diagram (9.18), the map $i_{\mathbb{R}}$ is an isometry and $p'_{U, \mathbb{R}}$ and $p'_{V, \mathbb{R}}$ are co-isometries.

When W decreases to $\{0\}$, V decreases to $\hat{i}(U)$ and $V_{\mathbb{R}}$ decreases to $\hat{i}(U)_{\mathbb{R}}$. More precisely, if $(W_n)_{n \in \mathbb{N}}$ is a defining sequence in $\mathcal{U}(\hat{G})$ and if $V_n := \hat{i}(U) + \hat{s}(W_n)$, then

$$\bigcap_{n \in \mathbb{N}} V_{n, \mathbb{R}} = \hat{i}(U)_{\mathbb{R}}.$$

This shows that $\|i_{UW, \mathbb{R}}(e)\|_{\pi_* \overline{F}_V}$ is a non-increasing function of $W \in \mathcal{U}(\hat{G})$. Moreover the equality (9.20) follows from Proposition 9.2.5 applied to $H = F_{\mathbb{R}}^{\text{Hilb}}$ and $H_n := V_{n, \mathbb{R}} \cap F_{\mathbb{R}}^{\text{Hilb}}$.

2) In the commutative diagram (9.18), the maps $p'_{V, \mathbb{R}}$, $q_{\mathbb{R}}$ and $p'_{W, \mathbb{R}}$ are co-isometries. Therefore $q_{UW, \mathbb{R}}$ is a co-isometry. □

We shall denote by \overline{E}_U^W the hermitian vector bundle over $\text{Spec } \mathcal{O}_K$ defined by the \mathcal{O}_K -modules E_U equipped with the hermitian structure induced by the one of \overline{F}_V . In other words, the hermitian structure on \overline{E}_U^W is defined in a way that makes the second line of (9.17) an admissible short exact sequence of hermitian vector bundles over $\text{Spec } \mathcal{O}_K$:

$$(9.21) \quad 0 \longrightarrow \overline{E}_U^W \xrightarrow{i_{UW}} \overline{F}_V \xrightarrow{q_{UW}} \overline{G}_W \longrightarrow 0.$$

(Recall that, according to Lemma 9.2.4, 2), the map $q_{UW, \mathbb{R}}$ is a co-isometry.)

Lemma 9.2.6. *For any (U, W) in $\mathcal{U}(\hat{E}) \times \mathcal{U}(\hat{G})$, we have:*

$$(9.22) \quad h_\theta^0(\overline{F}_{\hat{i}(U) + \hat{s}(W)}) \leq h_\theta^0(\overline{E}_U^W) + h_\theta^0(\overline{G}_W).$$

Moreover, for any U in $\mathcal{U}(\hat{E})$, $h_\theta^0(\overline{E}_U^W)$ is a non-decreasing function of $W \in \mathcal{U}(\hat{G})$ and

$$(9.23) \quad \lim_{W \in \mathcal{U}(\hat{G})} h_\theta^0(\overline{E}_U^W) = h_\theta^0(\overline{E}_U).$$

Proof. The estimate (9.22) follows the subadditivity of h_θ^0 for admissible short exact sequences of hermitian vector bundles (see Proposition 3.8.1) applied to (9.21).

The second assertion in Lemma 9.2.6 follows from part 1) of Lemma 9.2.4. □

Lemma 9.2.7. *For any (U, W) in $\mathcal{U}(\widehat{E}) \times \mathcal{U}(\widehat{G})$, we have*

$$(9.24) \quad h_\theta^1(\overline{F}_{\hat{i}(U)+\hat{s}(W)}) \leq h_\theta^1(\overline{E}_U^W) + h_\theta^1(\overline{G}_W).$$

and

$$(9.25) \quad h_\theta^1(\overline{E}_U^W) \leq h_\theta^1(\overline{E}_U).$$

Proof. The estimate (9.24) follows from the subadditivity of h_θ^1 for admissible short exact sequences of hermitian vector bundles (see Proposition 3.8.1 and equation (3.55)) applied to (9.21).

The estimate (9.19) shows that the identity map $Id_{E_U} : \overline{E}_U \rightarrow \overline{E}_U^W$ is norm decreasing. This implies (9.25) (see Proposition 3.3.2, 2)). \square

9.2.3. Proof of Theorem 9.2.1: II. Completion of the proof. We keep the notation of the previous paragraph.

1) (i) Let P be a finitely generated \mathcal{O}_K -submodule of $\widehat{F} \cap F_{\mathbb{R}}^{\text{Hilb}}$.

According to Lemma (9.2.3), the inverse images of P by \hat{i} and by $i_{\mathbb{R}}$ coincide and define a (finitely generated) \mathcal{O}_K -submodule of $\widehat{E} \cap E_{\mathbb{R}}^{\text{Hilb}}$.

Similarly, by the commutativity of (9.14), the images of P by \hat{q} and by $q_{\mathbb{R}}$ coincide and define a (finitely generated) \mathcal{O}_K -submodule of $\widehat{G} \cap G_{\mathbb{R}}^{\text{Hilb}}$.

Let $\overline{i^{-1}(P)}$, \overline{P} and $\overline{q(P)}$ the hermitian vector bundles over $\text{Spec } \mathcal{O}_K$ defines by $i^{-1}(P)$, P and $q(P)$ equipped with the restrictions of the hermitian structures on \widehat{E} , \widehat{F} , and \widehat{G} . Let $\overline{q(P)}^{\text{quot}}$ denotes the hermitian vector bundle over $\text{Spec } \mathcal{O}_K$ defined by $q(P)$ equipped with the hermitian structures quotient, via the map $q : P \rightarrow q(P)$, of the ones on \overline{P} .

By construction,

$$(9.26) \quad 0 \rightarrow \overline{i^{-1}(P)} \xrightarrow{i} \overline{P} \xrightarrow{q} \overline{q(P)}^{\text{quot}} \rightarrow 0$$

is an admissible short exact sequence of hermitian vector bundles over $\text{Spec } \mathcal{O}_K$, and accordingly, by Proposition 3.8.1, we have

$$(9.27) \quad h_\theta^0(\overline{P}) \leq h_\theta^0(\overline{i^{-1}(P)}) + h_\theta^0(\overline{q(P)}^{\text{quot}}).$$

Besides the map $q : \overline{P} \rightarrow \overline{q(P)}$ has operator norms ≤ 1 , and therefore the identity map $Id_{q(P)} : \overline{q(P)}^{\text{quot}} \rightarrow \overline{q(P)}$ also. Accordingly, by Proposition 3.3.2, 1), we have

$$(9.28) \quad h_\theta^0(\overline{q(P)}^{\text{quot}}) \leq h_\theta^0(\overline{q(P)}).$$

From (9.27) and (9.28), we derive:

$$\begin{aligned} h_\theta^0(\overline{P}) &\leq h_\theta^0(\overline{i^{-1}(P)}) + h_\theta^0(\overline{q(P)}) \\ &\leq \underline{h}_\theta^0(\widehat{E}) + \underline{h}_\theta^0(\widehat{G}). \end{aligned}$$

This proves (9.7).

(ii) Let $(U_j)_{j \in \mathbb{N}}$ (resp. $(W_k)_{k \in \mathbb{N}}$) be a defining filtration $\mathcal{U}(\widehat{E})$ (resp. in $\mathcal{U}(\widehat{G})$) such that

$$(9.29) \quad \lim_{j \rightarrow +\infty} h_\theta^0(\overline{E}_{U_j}) = \overline{h}_\theta^0(\widehat{E})$$

and

$$(9.30) \quad \lim_{k \rightarrow +\infty} h_\theta^0(\overline{E}_{W_k}) = \overline{h}_\theta^0(\widehat{G}).$$

Let us choose ϵ a positive real number. There exists j_0 (resp. k_0) in \mathbb{N} such that, for any integer $j \geq j_0$ (resp. $k \geq k_0$), we have

$$(9.31) \quad h_\theta^0(\overline{E}_{U_j}) \leq \overline{h}_\theta^0(\widehat{E}) + \epsilon/3$$

and

$$(9.32) \quad h_\theta^0(\overline{G}_{W_k}) \leq \overline{h}_\theta^0(\widehat{G}) + \epsilon/3.$$

Moreover, according to (9.23), for any $j \in \mathbb{N}$, there exists $k(j) \in \mathbb{N}$ such that, for any $k \in \mathbb{N}_{\geq k(j)}$,

$$(9.33) \quad h_\theta^0(\overline{E}_{U_j}^{W_k}) \leq h_\theta^0(\widehat{E}_{U_j}) + \epsilon/3.$$

Together with (9.22), these three estimates (9.31)–(9.33) show that, for any $(j, k) \in \mathbb{N}^2$,

$$j \geq j_0 \text{ and } k \geq \max(k_0, k(j)) \implies h_\theta^0(\overline{F}_{\hat{i}(U_j) + \hat{s}(W_k)}) \leq h_\theta^0(\overline{E}) + h_\theta^0(\overline{G}) + \epsilon.$$

Since the submodules $\hat{i}(U_j) + \hat{s}(W_k)$ constitute a basis of neighborhoods of 0 in $\mathcal{U}(\overline{F})$, this shows that

$$\liminf_{V \in \mathcal{U}(\widehat{F})} h_\theta^0(\overline{F}_V) \leq h_\theta^0(\overline{E}) + h_\theta^0(\overline{G}) + \epsilon.$$

As ϵ is arbitrary in \mathbb{R}_+^* , this establishes (9.8).

(iii) Let us recall that $h_\theta^1(\widehat{E}_U)$ (resp. $h_\theta^1(\widehat{G}_W)$) is a non-increasing function of U in $\mathcal{U}(\widehat{E})$ (resp. of W in $\mathcal{U}(\widehat{G})$) and that

$$(9.34) \quad h_\theta^1(\overline{E}) = \lim_{U \in \mathcal{U}(\widehat{E})} h_\theta^1(\overline{E}_U) \text{ (resp. } h_\theta^1(\overline{G}) = \lim_{W \in \mathcal{U}(\widehat{G})} h_\theta^1(\overline{G}_W)).$$

Similarly $h_\theta^1(\overline{F}_{\hat{i}(U) + \hat{s}(W)})$ is a non-increasing function of U in $\mathcal{U}(\widehat{E})$ and W in $\mathcal{U}(\widehat{G})$, and

$$(9.35) \quad h_\theta^1(\overline{F}) = \lim_{U \in \mathcal{U}(\widehat{E}), V \in \mathcal{U}(\widehat{G})} h_\theta^1(\overline{F}_{\hat{i}(U) + \hat{s}(V)}).$$

The estimates (9.24) and (9.25) show that, for any (U, W) in $\mathcal{U}(\widehat{E}) \times \mathcal{U}(\widehat{G})$,

$$(9.36) \quad h_\theta^1(\overline{F}_{\hat{i}(U) + \hat{s}(W)}) \leq h_\theta^1(\overline{E}_U) + h_\theta^1(\overline{G}_W).$$

The estimate (9.9)

$$h_\theta^1(\widehat{F}) \leq h_\theta^1(\widehat{E}) + h_\theta^1(\widehat{G})$$

directly follows from this estimate, together with the expressions (9.34) and (9.35) for $h_\theta^1(\widehat{E})$, $h_\theta^1(\widehat{G})$, and $h_\theta^1(\widehat{F})$.

2) Let us now assume that \widehat{E} is a hermitian vector bundle \overline{E} .

Then $i(E)$ is contained in $\widehat{F} \cap F_{\mathbb{R}}^{\text{Hilb}}$. Therefore, with the notation of 1) (i), $\underline{h}_\theta^0(\overline{F})$ is the supremum of the $h_\theta^0(\overline{P})$, where P denotes a finitely generated \mathcal{O}_K -submodule of $\widehat{F} \cap F_{\mathbb{R}}^{\text{Hilb}}$ containing $i(P)$.

For any such P , its image $q(P)$ is contained in $\widehat{G} \cap G_{\mathbb{R}}^{\text{Hilb}}$ and the admissible short exact sequence (9.26) takes the form:

$$(9.37) \quad 0 \longrightarrow \overline{E} \xrightarrow{i} \overline{P} \xrightarrow{q} \overline{q(P)} \longrightarrow 0.$$

Conversely, for any finitely generated \mathcal{O}_K -submodule Q of $\widehat{G} \cap G_{\mathbb{R}}^{\text{Hilb}}$, its inverse image by q , $P := q^{-1}(Q)$, is a finitely generated \mathcal{O}_K -submodule $\widehat{F} \cap F_{\mathbb{R}}^{\text{Hilb}}$ containing $i(P)$ such that $Q = q(P)$.

The inequality (4.25) (or equivalently (4.28)) applied to (9.37) takes the form:

$$h_\theta^0(\overline{q(P)}) \leq h_\theta^0(\overline{P}) - \widehat{\deg} \pi_* \overline{E}.$$

From this inequality, by taking the supremum over the submodules P as above, we obtain (9.10).

3) Let us now assume that \widehat{G} is a hermitian vector bundle \overline{G} .

For any $U \in \mathcal{U}(\widehat{E})$, from the admissible short exact sequence of pro-hermitian vector bundles

$$0 \longrightarrow \widehat{E} \xrightarrow{i} \widehat{F} \xrightarrow{q} \widehat{G} \longrightarrow 0,$$

we derive an admissible short exact sequence of hermitian vector bundles:

$$(9.38) \quad 0 \longrightarrow \overline{E}_U \longrightarrow \widehat{F}_{i(U)} \longrightarrow \widehat{G} \longrightarrow 0.$$

From (9.38), we derive the relations

$$h_\theta^0(\overline{E}_U) \leq h_\theta^0(\widehat{F}_{i(U)})$$

(by Proposition 3.3.2, 1)) and

$$h_\theta^0(\widehat{F}_{i(U)}) - h_\theta^1(\widehat{F}_{i(U)}) = h_\theta^0(\overline{E}_U) - h_\theta^1(\overline{E}_U) + \widehat{\deg} \pi_* \overline{G}$$

(by the additivity of the Arakelov degree (2.12) and the Poisson-Riemann-Roch formula (3.11)).

Thus we obtain:

$$h_\theta^1(\overline{E}_U) \leq h_\theta^1(\widehat{F}_{i(U)}) + \widehat{\deg} \pi_* \overline{G},$$

and the estimate (9.11) follows by taking the limit when $U \in \mathcal{U}(\widehat{E})$ shrinks to $\{0\}$.

9.3. Short exact sequences and strongly summable pro-hermitian vector bundles.

9.3.1. In this section, we establish the following permanence properties of strongly summable pro-hermitian vector bundles, which may be seen as refinements of Corollary 9.2.2:

Theorem 9.3.1. *Consider an admissible short exact sequence of pro-hermitian vector bundles over the arithmetic curve $\text{Spec } \mathcal{O}_K$:*

$$(9.39) \quad 0 \longrightarrow \widehat{E} \xrightarrow{i} \widehat{F} \xrightarrow{q} \widehat{G} \longrightarrow 0$$

1) When \widehat{E} has finite rank, \widehat{F} is strongly summable if and only if \widehat{G} is strongly summable.

2) When \widehat{G} has finite rank, \widehat{F} is strongly summable if and only if \widehat{E} is strongly summable.

With the notation of Theorem 9.3.1, for any $\delta \in \mathbb{R}$, the diagram

$$(9.40) \quad 0 \longrightarrow \widehat{E} \otimes \mathcal{O}(\delta) \xrightarrow{i} \widehat{F} \otimes \mathcal{O}(\delta) \xrightarrow{q} \widehat{G} \otimes \mathcal{O}(\delta) \longrightarrow 0$$

is again an admissible short exact sequence of pro-hermitian vector bundles over $\text{Spec } \mathcal{O}_K$. Therefore, by the very definition of a θ -finite pro-hermitian vector bundle, Theorem 9.3.1 implies:

Corollary 9.3.2. 1) When \widehat{E} has finite rank, \widehat{F} is θ -finite if and only if \widehat{G} is θ -finite.

2) When \widehat{G} has finite rank, \widehat{F} is θ -finite if and only if \widehat{E} is θ -finite. □

9.3.2. *Proof of Theorem 9.3.1: I. Preliminary.* The following lemma gathers various facts that will be needed in the proof of part 1) of Theorem 9.3.1.

Lemma 9.3.3. *Consider an admissible short exact sequence of pro-hermitian vector bundles over $\text{Spec } \mathcal{O}_K$,*

$$0 \longrightarrow \overline{E} \xrightarrow{i} \widehat{F} \xrightarrow{q} \widehat{G} \longrightarrow 0,$$

with \overline{E} of finite rank (or, equivalently, a hermitian vector bundle over $\text{Spec } \mathcal{O}_K$).

Let $(V_k)_{k \in \mathbb{N}}$ be a defining sequence in $\mathcal{U}(\widehat{F})$ and let $(W_k)_{k \in \mathbb{N}} := (\hat{q}(V_k))_{k \in \mathbb{N}}$ be its image by \hat{q} .

1) There exists $k_0 \in \mathbb{N}$ such that, for any $k \in \mathbb{N}_{\geq k_0}$,

$$V_k \cap \hat{i}(E) = \{0\}$$

and the image of $\hat{i}(E)$ in \hat{F}/V_k is saturated. Moreover, for any such integer k_0 , the sequence $(W_k)_{k \geq k_0}$ is a defining sequence in $\mathcal{U}(\hat{G})$.

2) For any $k \in \mathbb{N}$, let us consider the hermitian vector bundle $\overline{F}_k := \overline{F}_{V_k}$ and the admissible surjective morphism

$$r_k := p_{V_k V_{k+1}} : \overline{F}_{k+1} \longrightarrow \overline{F}_k.$$

Similarly, for any $k \in \mathbb{N}_{\geq k_0}$, let us consider $\overline{G}_k := \overline{G}_{W_k}$ and

$$s_k := p_{W_k W_{k+1}} : \overline{G}_{k+1} \longrightarrow \overline{G}_k.$$

For any $k \in \mathbb{N}_{\geq k_0}$, the map \hat{q} induces a map

$$q_k : F_k := \hat{F}/V_k \longrightarrow G_k := \hat{G}/W_k,$$

which fits into an exact sequence:

$$0 \longrightarrow E \xrightarrow{i_k} F_k \xrightarrow{q_k} G_k \longrightarrow 0,$$

where $i_k := p_{V_k} \circ i$.

Moreover, q_k belongs to $\text{Hom}_{\mathcal{O}_K}^{\leq 1}(\overline{F}_k, \overline{G}_k)$.

3) For any $k \in \mathbb{N}_{\geq k_0}$, the map q_k defines an isomorphism

$$\phi_k := q_k|_{\ker r_k} : \ker r_k \xrightarrow{\sim} \ker s_k.$$

Moreover ϕ_k belongs to $\text{Hom}_{\mathcal{O}_K}^{\leq 1}(\overline{\ker r_k}, \overline{\ker s_k})$ and, for any $\eta \in \mathbb{R}_+^*$, there exists $k(\eta) \in \mathbb{N}_{\geq k_0}$ such that, for any $k \in \mathbb{N}_{\geq k(\eta)}$, ϕ_k^{-1} belongs to $\text{Hom}_{\mathcal{O}_K}^{\leq 1}(\overline{\ker s_k}, \overline{\ker r_k} \otimes \mathcal{O}(\eta))$.

The proof of the last part of Lemma 9.3.3 will rely on the following proposition concerning the geometry of sub-quotients of Hilbert spaces.

Proposition 9.3.4. *Let $(H, \|\cdot\|)$ be a (real of complex) Hilbert space and let*

$$H_0 \supseteq H_1 \supseteq H_2 \supseteq H_3 \supseteq \dots \supseteq H_n \supseteq H_{n+1} \dots$$

be a decreasing sequence of closed vector subspaces of H such that

$$\bigcap_{n \in \mathbb{N}} H_n = \{0\}.$$

Let P be a (necessarily closed) finite dimensional vector subspace of H .

1) *There exists $n_0 \in \mathbb{N}$ such that, for any $n \geq n_0$,*

$$H_n \cap P = \{0\}.$$

Then, for any such integer $n \geq n_0$, the linear map

$$\phi_n : H_n/H_{n+1} \longrightarrow (H_n + P)/(H_{n+1} + P),$$

induced by the inclusion $H_n \hookrightarrow H_n + P$, is bijective.

2) *For every $n \in \mathbb{N}$, let us equip H_n/H_{n+1} (resp. $(H_n + P)/(H_{n+1} + P)$) with the norm quotient of the norm $\|\cdot\|$ on H_n (resp. on $H_n + P$)³². Then, for any $n \in \mathbb{N}$,*

$$(9.41) \quad \|\phi_n\| := \sup_{x \in H_n/H_{n+1}, \|x\| \leq 1} \|\phi_n(x)\| \leq 1.$$

Moreover the sequence $(\|\phi_n^{-1}\|)_{n \geq n_0}$, defined by

$$\|\phi_n^{-1}\| := \sup_{x \in (H_n + P)/(H_{n+1} + P), \|x\| \leq 1} \|\phi_n^{-1}(x)\|,$$

³²This is indeed a norm since H_{n+1} (resp. $H_{n+1} + P$) is closed in H_n (resp. $H_n + P$).

satisfies

$$(9.42) \quad \lim_{n \rightarrow \infty} \|\phi_n^{-1}\| = 1.$$

Proof of Proposition 9.3.4. Assertion 1) is clear, and the upper bound (9.41) also.

For every $n \in \mathbb{N}$, let us denote by

$$\pi_n : H \longrightarrow H_n$$

the orthogonal projection onto H_n , and let $p : H \longrightarrow P$ denote the orthogonal projection onto P .

Since $\bigcap_{n \in \mathbb{N}} H_n = \{0\}$, for every $x \in H$,

$$\lim_{n \rightarrow +\infty} \|\pi_n(x)\| = 0.$$

As P is finite dimensional, the operator norm of the restriction to P of π_n ,

$$\|\pi_n|_P\| := \max_{x \in P, \|x\| \leq 1} \|\pi_n(x)\|,$$

converges also to 0 when n goes to $+\infty$.

The restriction to H_n of p ,

$$p|_{H_n} : H_n \longrightarrow P$$

is the adjoint of

$$\pi_n|_P : P \longrightarrow H_n,$$

and therefore its operator norm

$$\|p|_{H_n}\| = \|\pi_n|_P\|$$

converges also to 0. This easily implies (9.42). \square

Proof of Lemma 9.3.3. Assertion 1) easily follows from Propositions 5.4.1 and 5.4.4.

Assertion 2) is a straightforward consequence of the definitions.

Assertion 3) follows from Proposition 9.3.4 applied to $H = F_\sigma^{\text{Hilb}}$, $H_n = V_{n,\sigma} \cap F_\sigma^{\text{Hilb}}$, and $P = i_\sigma(E_\sigma)$. \square

9.3.3. Proof of Theorem 9.3.1: II. Completion of the proof.

1) Let us assume that \widehat{E} has finite rank or, in other words, that it is defined by some hermitian vector bundle \overline{E} over $\text{Spec } \mathcal{O}_K$. Then the conclusions of Lemma 9.3.3 hold, and we shall use its notation.

From Part 3) of Lemma 9.3.3, it follows that, for any $\epsilon \in \mathbb{R}$ and any $k \in \mathbb{N}_{k \geq k_0}$,

$$(9.43) \quad h_\theta^0(\overline{\ker r_k} \otimes \overline{\mathcal{O}(\epsilon)}) \leq h_\theta^0(\overline{\ker s_k} \otimes \overline{\mathcal{O}(\epsilon)})$$

and that, for any $(\epsilon, \epsilon') \in \mathbb{R}^2$ and any $k \in \mathbb{N}$,

$$(9.44) \quad \epsilon' < \epsilon \text{ and } k \geq k(\epsilon - \epsilon') \implies h_\theta^0(\overline{\ker s_k} \otimes \overline{\mathcal{O}(\epsilon')}) \leq h_\theta^0(\overline{\ker r_k} \otimes \overline{\mathcal{O}(\epsilon)}).$$

Let us assume that \widehat{F} is strongly summable. Then \widehat{F} satisfies **StS₂** and we may choose a defining sequence $(V_k)_{k \in \mathbb{N}}$ in $\mathcal{U}(\widehat{F})$ and $\epsilon > 0$ such that

$$\sum_{k \in \mathbb{N}} h_\theta^0(\overline{\ker r_k} \otimes \overline{\mathcal{O}(\epsilon)}) < +\infty.$$

For any $\epsilon' \in]0, \epsilon[$, the upper bound (9.44) shows that

$$\sum_{k \in \mathbb{N}} h_\theta^0(\overline{\ker s_k} \otimes \overline{\mathcal{O}(\epsilon')}) < +\infty.$$

Consequently \widehat{G} also satisfies **StS₂** and is therefore strongly summable.

Conversely, let us assume that \widehat{G} is strongly summable, and let us choose a defining sequence $(W_k)_{k \in \mathbb{N}}$ in $\mathcal{U}(\widehat{G})$ and $\epsilon \in \mathbb{R}_+^*$ such that, if we let $\overline{G}_k := \overline{G}_{W_k}$, the admissible surjective morphisms

$$s_k := p_{W_k W_{k+1}} : \overline{G}_{k+1} \longrightarrow \overline{G}_k$$

satisfy:

$$\sum_{k \in \mathbb{N}} h_\theta^0(\overline{\ker s_k} \otimes \overline{\mathcal{O}(\epsilon)}) < +\infty.$$

There exists a continuous \mathcal{O}_K -linear splitting $\hat{s} : \widehat{G} \longrightarrow \widehat{F}$ of the short exact sequence

$$0 \longrightarrow \widehat{E} \xrightarrow{\hat{i}} \widehat{F} \xrightarrow{\hat{q}} \widehat{G} \longrightarrow 0$$

(sees 9.1.1, Remark (ii)), and we may define, for every $k \in \mathbb{N}$,

$$V_k := \hat{s}(W_k).$$

Then $W_k = \hat{q}(V_k)$, and we may apply Lemma 9.3.3. Notably, the estimate (9.43) imply that

$$\sum_{k \in \mathbb{N}} h_\theta^0(\overline{\ker r_k} \otimes \overline{\mathcal{O}(\epsilon)}) \leq \sum_{k \in \mathbb{N}} h_\theta^0(\overline{\ker s_k} \otimes \overline{\mathcal{O}(\epsilon)}) < +\infty.$$

Therefore satisfies **StS₂** and is strongly summable.

2) The morphism i belongs to $\text{Hom}_{\mathcal{O}_K}^{\leq 1}(\widehat{E}, \widehat{F})$ and $\hat{i} : \widehat{E} \longrightarrow \widehat{F}$ is a strict morphism of topological \mathcal{O}_K -modules. Consequently, according to Proposition 8.7.4, if \widehat{F} is strongly summable, then \widehat{E} is strongly summable.

Conversely, let us assume that \widehat{E} is strongly summable and that \widehat{G} has finite rank, or equivalently, that \widehat{G} is defined by some hermitian vector bundle \overline{G} over $\text{Spec } \mathcal{O}_K$.

Then \widehat{E} satisfies **StS₂** and we may choose a defining sequence $(U_k)_{k \in \mathbb{N}}$ in $\mathcal{U}(\widehat{E})$ and $\epsilon > 0$ such that, if we let $\overline{E}_k := \overline{E}_{U_k}$, the admissible surjective morphisms

$$q_k := p_{U_k U_{k+1}} : \overline{E}_{k+1} \longrightarrow \overline{E}_k$$

satisfy:

$$\sum_{k \in \mathbb{N}} h_\theta^0(\overline{\ker q_k} \otimes \overline{\mathcal{O}(\epsilon)}) < +\infty.$$

Let us consider the sequence $(V_k)_{k \in \mathbb{N}} := (\hat{i}(U_k))_{k \in \mathbb{N}}$. As

$$0 \longrightarrow \widehat{E} \xrightarrow{\hat{i}} \widehat{F} \xrightarrow{\hat{q}} G \longrightarrow 0$$

is a strict short exact sequence in $CTC_{\mathcal{O}_K}$ and G is a discrete \mathcal{O}_K -module, it is a defining sequence in $\mathcal{U}(\widehat{F})$.

Moreover, if we let $\overline{F}_k := \overline{F}_{V_k}$ and if we introduce the admissible surjective morphisms

$$r_k := p_{V_k V_{k+1}} : \overline{F}_{k+1} \longrightarrow \overline{F}_k,$$

it is straightforward that the morphisms

$$i_k : E_k := \widehat{E}/V_k \longrightarrow F_k/U_k$$

induced by \hat{i} fit into commutative diagrams of \mathcal{O}_K -module with exact lines:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_{k+1} & \xrightarrow{i_{k+1}} & F_{k+1} & \longrightarrow & G & \longrightarrow & 0 \\ & & \downarrow q_k & & \downarrow r_k & & \downarrow Id_G & & \\ 0 & \longrightarrow & E_k & \xrightarrow{i_k} & F_k & \longrightarrow & G & \longrightarrow & 0. \end{array}$$

Therefore, for any $k \in \mathbb{N}$, i_{k+1} defines an isomorphism

$$i_{k+1} : \ker q_k \longrightarrow \ker r_k,$$

that is easily seen to be an isometric isomorphism of hermitian vector bundles:

$$i_{k+1} : \overline{\ker q_k} \xrightarrow{\sim} \overline{\ker r_k}.$$

Consequently,

$$h_\theta^0(\overline{\ker r_k} \otimes \overline{\mathcal{O}(\epsilon)}) = h_\theta^0(\overline{\ker q_k} \otimes \overline{\mathcal{O}(\epsilon)}).$$

Finally,

$$\sum_{k \in \mathbb{N}} h_\theta^0(\overline{\ker r_k} \otimes \overline{\mathcal{O}(\epsilon)}) = \sum_{k \in \mathbb{N}} h_\theta^0(\overline{\ker q_k} \otimes \overline{\mathcal{O}(\epsilon)}) < +\infty$$

and \widehat{F} also satisfies **StS₂**.

APPENDIX A. LARGE DEVIATIONS AND CRAMÉR'S THEOREM

In this Appendix, we present some basic results in the theory of large deviations in a form suited to the application to euclidean lattices discussed in Section 4.4.

In particular, we formulate a general version of Cramér's theory of large deviation (Theorem A.3.1). Recall that, a measurable function

$$H : \mathcal{E} \longrightarrow \mathbb{R}$$

being given on some probability space $(\mathcal{E}, \mathcal{T}, \mu)$, this theory describes the asymptotic behavior, when the positive integer n goes to infinity, of the values of the function

$$H_n : \mathcal{E}^n \longrightarrow \mathbb{R}$$

defined by

$$(A.1) \quad H_n(e_1, \dots, e_n) := H(e_1) + \dots + H(e_n).$$

This description is formulated in terms of the integral

$$\log \int_X e^{\xi H} d\mu,$$

as a function of $\xi \in \mathbb{R}$ with values in $] -\infty, +\infty]$, and of its Legendre-Fenchel transform.

Actually, for the application to euclidean lattices of the theory of large deviations, we rely on some extension of Cramér's theorem covering the situation where μ is an arbitrary σ -finite positive measure, assuming that H is non-negative.³³

Such extensions of Cramér's theorem are possibly known to some experts, but for lack of references, in Sections A.4 and A.5, we formulate and establish suitable versions of Cramér's theorem (Theorems A.4.4 and A.5.1) covering the case where $\mu(\mathcal{E})$ is possibly $+\infty$. We achieve this by a simple reduction to the case where μ is a probability measure.

In the first two sections (A.1 and A.2) of this Appendix, we also extend various preliminary results concerning the asymptotic behavior of the values of H_n on \mathcal{E}^n when n goes to infinity to this more general setting where $\mu(\mathcal{E})$ is possibly $+\infty$. Here again, for lack of suitable references, we have included some details concerning these “well-known” results. Then, in Section A.3, we recall diverse forms of the “classical” Cramér's theorem, concerning the situation where μ is a probability measure.

Let us finally point out that the generalized version of Cramér's theorem presented in this Appendix has close relations to the so-called *canonical ensembles* in statistical mechanics and with the

³³Indeed, in Section 4.4, to investigate the properties of the “asymptotic invariants” $\tilde{h}_{\text{Ar}}^0(\overline{E}, t)$ attached to some euclidean lattice $\overline{E} := (E, \|\cdot\|)$, we consider the situation where \mathcal{E} is the free \mathbb{Z} -module E underlying \overline{E} , where μ is the counting measure $\sum_{e \in E} \delta_e$, and where H is the squared euclidean norm $\|\cdot\|^2$.

existence of thermodynamic limits. We do not discuss this in details, but we simply refer the reader to [Khi49] and [Sch62] for presentations of statistical thermodynamics that emphasize the points of contact between statistical mechanics and limit theorems in probability³⁴, and we indicate that the notation in Section A.5 have been chosen to express these relations.

The first paragraph A.5.1 of the final Section A.5 of this Appendix summarizes some of its main results in a form suitable for their applications in Section 4.4. Section A.5 has been written to be accessible without a knowledge of the formalism previously developed in this Appendix, and could be read immediately after this introduction.

A.1. Notation and preliminaries.

A.1.1. *Notation.* In this Appendix, we consider a measure space $(\mathcal{E}, \mathcal{T}, \mu)$ defined by a set \mathcal{E} , a σ -algebra \mathcal{T} over \mathcal{E} , and a non-zero σ -finite non-negative measure $\mu : \mathcal{T} \rightarrow [0, +\infty]$.

Besides, we consider a \mathcal{T} -measurable function

$$H : \mathcal{E} \rightarrow \mathbb{R}.$$

We shall denote by $\inf_{\mu} H$ (resp. $\sup_{\mu} H$) its essential infimum (resp. supremum) with respect to the measure μ .

For every positive integer n , we may introduce the n -th power of the measure space $(\mathcal{E}, \mathcal{T}, \mu)$ — it is defined as the product \mathcal{E}^n of n copies of \mathcal{E} , equipped with the σ -algebra

$$\mathcal{T}^{\otimes n} := \mathcal{T} \otimes \cdots \otimes \mathcal{T} \text{ (} n\text{-times)}$$

over \mathcal{E}^n and with the product σ -finite measure

$$\mu^{\otimes n} := \mu \otimes \cdots \otimes \mu \text{ (} n\text{-times)}$$

on $\mathcal{T}^{\otimes n}$ — and we may consider the function $H_n : \mathcal{E}^n \rightarrow \mathbb{R}$ defined by (A.1).

A.1.2. *Log-Laplace transform.* Recall that a function $p : \mathbb{R} \rightarrow]-\infty, +\infty]$ is lower semi-continuous and convex if and only if

$$\text{Gr}^{\geq}(p) := \{(x, y) \in \mathbb{R}^2 \mid y \geq p(x)\}$$

is a closed convex subset of \mathbb{R}^2 . One easily shows that this holds precisely when there exists an interval $I \subset \mathbb{R}$ such that the following conditions are satisfied:

- (1) $p|_I$ is real valued, continuous and convex;
- (2) $p_{\mathbb{R} \setminus I} \equiv +\infty$;
- (3) if $I \neq \emptyset$ and if $a := \inf I \notin I \cup \{-\infty\}$, then $\lim_{x \rightarrow a+} p(x) = +\infty$;
if $I \neq \emptyset$ and if $b := \sup I \notin I \cup \{+\infty\}$, then $\lim_{x \rightarrow b-} p(x) = +\infty$.

If $p : \mathbb{R} \rightarrow]-\infty, +\infty]$ is non-increasing and convex, then there exists a unique $c \in [-\infty, +\infty]$ such that, for any $x \in \mathbb{R}$,

$$x < c \implies p(x) \in \mathbb{R} \text{ and } x > c \implies p(x) = +\infty,$$

and p is lower semi-continuous if and only if

$$\lim_{x \rightarrow c-} p(x) = p(c).$$

Similar remarks apply *mutatis mutandis* to non-decreasing convex functions, and to concave functions from \mathbb{R} to $[-\infty, +\infty[$.

The following proposition is a straightforward consequence of the basic results of Lebesgue integration theory:

³⁴See also [Lan73] and [Ell85] for related mathematical results and references.

Proposition A.1.1. *For every $\xi \in \mathbb{R}$, the integral $\int_X \exp(\xi H) d\mu$ belongs to $]0, +\infty]$ and the function*

$$\ell : \mathbb{R} \longrightarrow]-\infty, +\infty]$$

defined by

$$\ell(\xi) := \log \int_X e^{\xi H} d\mu$$

is lower semi-continuous and convex.

Moreover the restriction $\ell|_{\mathring{I}}$ of ℓ to the interior \mathring{I} of the interval $I := \ell^{-1}(\mathbb{R})$ is real analytic. Unless H is constant μ -almost everywhere, it satisfies $\ell''|_{\mathring{I}} > 0$ and defines an increasing real analytic diffeomorphism $\ell' : \mathring{I} \xrightarrow{\sim} \ell'(\mathring{I})$ between the open intervals \mathring{I} and $\ell'(\mathring{I})$ in \mathbb{R} . \square

The function ℓ is appears in the litterature under various names. It is sometimes called the *log-Laplace transform* of the Borel measure $H_*\mu$ on \mathbb{R} , that is nothing but the “law” of H when μ is a probability measure. In this situation, it is also called the *logarithmic moment generating function* of $H_*\mu$ (in [Str11], for instance).

A.1.3. *The functions $A_n^>$ and $A_n^<$, and s_+ and s_- .* To every $x \in \mathbb{R}$, we may also attach the sequences $(A_{+,n}(x))_{n \geq 1}$ and $(A_{+,n}(x))_{n \geq 1}$ in $[0, +\infty]$ defined by:

$$A_n^>(x) = \mu^{\otimes n}(H_n^{-1}([nx, +\infty[))$$

and

$$A_n^<(x) = \mu^{\otimes n}(H_n^{-1}(]-\infty, nx])).$$

We shall be interested in situations where these sequences have an exponential asymptotic behavior, and accordingly, for every $x \in \mathbb{R}$, we consider the following elements of $[-\infty, +\infty]$:

$$(A.2) \quad s_+(x) := \sup_{n \geq 1} \frac{1}{n} \log A_n^>(x)$$

and

$$(A.3) \quad s_-(x) := \sup_{n \geq 1} \frac{1}{n} \log A_n^<(x).$$

Clearly, for any positive integer n , the function $A_n^> : \mathbb{R} \longrightarrow [0, +\infty]$ is non-increasing and the function $A_n^< : \mathbb{R} \longrightarrow [0, +\infty]$ is non-decreasing. Accordingly, the functions s_+ and $s_- : \mathbb{R} \longrightarrow [-\infty, +\infty]$ are respectively non-increasing and non-decreasing.

Observe also, that for any $x \in \mathbb{R}$, the following alternative holds: either (i) the function H is $< x$ μ -almost everywhere on \mathcal{E} , and then $A_n^>(x) = 0$ for every positive integer n ; or (ii) the function H takes values $\geq x$ on some subset of positive μ -measure, and then $A_n^>(x) > 0$ for every positive integer n .

When $x = \sup_\mu H$, either case may occur, but we always have:

$$A_n^>(\sup_\mu H) = \mu^{\otimes n}(H^{-1}(\sup_\mu H)^n) = \mu(H^{-1}(\sup_\mu H))^n,$$

and consequently:

$$s_+(\sup_\mu H) = \log \mu(H^{-1}(\sup_\mu H)).$$

A similar alternative holds for the sequence $(A_n^<(x))_{n \geq 1}$, and

$$s_-(\inf_\mu H) = \log \mu(H^{-1}(\inf_\mu H)).$$

A.2. Lanford's inequalities. In the present general framework (that does not require the finiteness of the measure μ and of the log-Laplace transform ℓ), the following inequalities play a crucial role in the study of the asymptotic behavior of the functions $A_n^>$ and $A_n^<$ when n goes to $+\infty$:

Lemma A.2.1. *For any (x, y) in \mathbb{R}^2 and any (p, q) in $\mathbb{N}_{\geq 1}^2$, we have:*

$$(A.4) \quad A_p^>(x) \cdot A_q^>(y) \leq A_{p+q}^>((px + qy)/(p + q))$$

and

$$(A.5) \quad A_p^<(x) \cdot A_q^<(y) \leq A_{p+q}^<((px + qy)/(p + q)).$$

In (A.4) and (A.5), we use the standard measure theoretical convention:

$$0.(+\infty) = (+\infty).0 = 0.$$

Together with the subadditivity and concavity arguments leading to the proofs of Proposition A.2.2 and Theorem A.2.5 *infra*, this lemma originates in Lanford's work [Lan73] on the rigorous derivation of “thermodynamic limits” in statistical mechanics.

Proof. Observe that the following inclusions hold between $\mathcal{T}^{\otimes n}$ -measurable subsets of \mathcal{E}^n :

$$(A.6) \quad H_p^{-1}([px, +\infty[) \times H_q^{-1}([qy, +\infty[) \subset H_{p+q}^{-1}([px + qy, +\infty[)$$

and

$$(A.7) \quad H_p^{-1}(]-\infty, px]) \times H_q^{-1}(]-\infty, qy]) \subset H_{p+q}^{-1}(]-\infty, px + qy]).$$

By applying the measure $\mu^{\otimes p+q}$ to both sides of (A.6) (resp., of (A.7)), we get the estimate (A.4) (resp. the estimate (A.5)). \square

When $y = x$, Lemma A.2.1 asserts that

$$(A.8) \quad A_p^>(x) A_p^>(x) \leq A_{p+q}^>(x)$$

and

$$(A.9) \quad A_p^<(x) A_p^<(x) \leq A_{p+q}^<(x).$$

In other words, the sequences $(\log A_n^>(x))_{n \geq 1}$ and $(\log A_n^<(x))_{n \geq 1}$ are superadditive.

Actually, combined with the alternative in A.1.3 above concerning the vanishing or the non-vanishing of the $A_n^>(x)$ and with the Lemma of Fekete on superadditive sequences (Lemma 4.4.3), the estimates (A.8) easily imply the following:

Proposition A.2.2. *For every $x \in \mathbb{R}$, precisely one of the following three conditions is satisfied:*

O[>] : *for every positive integer n , $A_n^>(x) = 0$;*

F[>] : *for every positive integer n , $A_n^>(x) \in]0, +\infty[$; then the sequence $((\log A_n^>(x))/n)_{n \geq 1}$ admits a limit:*

$$(A.10) \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \log A_n^>(x) = \sup_{n \geq 1} \frac{1}{n} \log A_n^>(x) =: s_+(x) \in]-\infty, +\infty].$$

I[>] : *for some positive integer n_0 , $A_{n_0}^>(x) = +\infty$. Then, for every integer $n \geq n_0$, $A_n^>(x) = +\infty$.*

A similar trichotomy holds with the sequence $(A_n^>(x))_{n \geq 1}$ replaced by $(A_n^<(x))_{n \geq 1}$, and $s_+(x)$ by $s_-(x)$, and defines conditions **O[<]**, **F[<]** and **I[<]**. \square

We may consider the subsets $I_{\mathbf{I}^>}$, $I_{\mathbf{F}^>}$ and $I_{\mathbf{O}^>}$ of \mathbb{R} defined by each the conditions **F[>]**, **I[>]** and **O[>]**. Clearly they constitute disjoint consecutive intervals, and we have:

$$\mathbb{R} = I_{\mathbf{I}^>} \cup I_{\mathbf{F}^>} \cup I_{\mathbf{O}^>}$$

and

$$s_+^{-1}(-\infty) = I_{\mathbf{O}^>} = \{x \in \mathbb{R} \mid \mu(H^{-1}([x, +\infty[)) = 0\}.$$

Similarly, we define disjoint consecutive intervals $I_{\mathbf{O}^<}$, $I_{\mathbf{F}^<}$ and $I_{\mathbf{I}^<}$ such that

$$\mathbb{R} = I_{\mathbf{O}^<} \cup I_{\mathbf{F}^<} \cup I_{\mathbf{I}^<}$$

and we have:

$$s_+^{-1}(-\infty) = I_{\mathbf{O}^<} = \{x \in \mathbb{R} \mid \mu(H^{-1}([-\infty, x])) = 0\}.$$

Lemma A.2.3. *For any two points x and y in $s_+^{-1}([-\infty, +\infty]) = I_{\mathbf{I}^>} \cup I_{\mathbf{F}^>}$ (resp., in $s_+^{-1}([-\infty, +\infty]) = I_{\mathbf{I}^<} \cup I_{\mathbf{F}^<}$) and any two positive rational number α and β such that $\alpha + \beta = 1$, we have:*

$$(A.11) \quad \begin{aligned} \alpha s_+(x) + \beta s_+(y) &\leq s_+(\alpha x + \beta y) \\ (\text{resp., } \alpha s_-(x) + \beta s_-(y) &\leq s_-(\alpha x + \beta y)). \end{aligned}$$

Proof. Consider positive a positive integer n such that $p := n\alpha$ and $q := n\beta$ are integers. Clearly p and q satisfy $p + q = n$, and from (A.4), by applying $(1/n) \log$, we derive:

$$\alpha \frac{1}{p} \log A_p^>(x) + \beta \frac{1}{q} \log A_q^>(y) \leq \frac{1}{n} \log A_n^>(\alpha x + \beta y).$$

The estimate (A.11) follows by taking the limit when n goes to infinity. \square

The inequality (A.11) implies that, if $s_+^{-1}(+\infty)$ is not empty, then $s_+^{-1}([-\infty, +\infty[)$ contains at most one point. When one investigates the asymptotic behavior of the numbers $(A_n^>(x))_{n \geq 1}$, it is sensible to exclude this case and to assume that the following condition is satisfied:

B[>] : For every $x \in \mathbb{R}$, $s_+(x) < +\infty$.

Clearly, this condition implies that $I_{\mathbf{I}^>}$ is empty.

A similar discussion applies to $s_-^{-1}(+\infty)$ and $I_{\mathbf{I}^<}$, and leads one to introduce the condition:

B[<] : For every $x \in \mathbb{R}$, $s_-(x) < +\infty$.

Lemma A.2.4. *The conditions **B[>]** and **B[<]** are satisfied if $\mu(\mathcal{E}) < +\infty$.*

*More generally, if there exists ξ in \mathbb{R}_+ (resp., in \mathbb{R}_-) such that $\ell(\xi) < +\infty$, then **B[>]** (resp., **B[<]**) is satisfied.*

Proof. When $\mu(\mathcal{E}) < +\infty$, $s_+(x)$ and $s_-(x)$ are bounded from above by $\log \mu(\mathcal{E})$ for every $x \in \mathbb{R}$.

Let us assume that ξ is an element of \mathbb{R}_+ such that $p(\xi) < +\infty$. Then, for every positive integer n , we have:

$$\int_{\mathcal{E}^n} e^{\xi H_n} d\mu^{\otimes n} = \left(\int_{\mathcal{E}} e^{\xi H} d\mu \right)^n = e^{np(\xi)} < +\infty.$$

Besides, for every $x \in \mathbb{R}$ and every $P \in H_n^{-1}([nx, +\infty[)$,

$$e^{\xi H_n(P)} \geq e^{n\xi x}.$$

Therefore

$$\mu^{\otimes n}(H_n^{-1}([nx, +\infty[)) e^{n\xi x} \leq e^{np(\xi)}$$

and

$$\frac{1}{n} \log A_n^>(x) \leq p(\xi) - \xi x.$$

This shows that

$$s_+(x) \leq p(\xi) - \xi x < +\infty.$$

The proof of **B[<]** when there exists ξ in \mathbb{R}_- such that $p(\xi) < +\infty$ is similar. \square

The following theorem summarizes and completes some of the results obtained so far concerning the asymptotic behavior of the functions $A_n^>$ and $A_n^<$:

Theorem A.2.5. *1) If Condition $\mathbf{B}^>$ is satisfied, then, for every $x \in \mathbb{R}$, either $\mu^{\otimes n}(H_n^{-1}([nx, +\infty[)) = 0$ for every positive integer n and $s_+(x) = -\infty$, or the sequence $(\log \mu^{\otimes n}(H_n^{-1}([nx, +\infty[)))_{n \geq 1}$ lies in \mathbb{R} , is superadditive, and satisfies:*

$$(A.12) \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \log \mu^{\otimes n}(H_n^{-1}([nx, +\infty[)) = s_+(x) \in \mathbb{R}.$$

The function $s_+ : \mathbb{R} \rightarrow [-\infty, +\infty[$ is non-increasing and concave.

Moreover, for any $x \in \mathbb{R}$,

$$x < \sup_{\mu} H \implies s_+(x) \in \mathbb{R} \quad \text{and} \quad x > \sup_{\mu} H \implies s_+(x) = -\infty,$$

and

$$s_+(\sup_{\mu} H) = \log \mu(H^{-1}(\sup_{\mu} H)).$$

2) Symmetrically, if Condition $\mathbf{B}^<$ is satisfied, then, for every $x \in \mathbb{R}$, either $\mu^{\otimes n}(H_n^{-1}(]-\infty, nx])) = 0$ for every positive integer n and $s_-(x) = -\infty$, or the sequence $(\log \mu^{\otimes n}(H_n^{-1}(]-\infty, nx])))_{n \geq 1}$ lies in \mathbb{R} , is superadditive, and satisfies:

$$(A.13) \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \log \mu^{\otimes n}(H_n^{-1}(]-\infty, nx])) = s_-(x) \in \mathbb{R}.$$

The function $s_- : \mathbb{R} \rightarrow [-\infty, +\infty[$ is non-decreasing and concave.

Moreover, for any $x \in \mathbb{R}$,

$$x < \inf_{\mu} H \implies s_-(x) = -\infty \quad \text{and} \quad x > \inf_{\mu} H \implies s_-(x) \in \mathbb{R},$$

and

$$s_-(\inf_{\mu} H) = \log \mu(H^{-1}(\inf_{\mu} H)).$$

Proof. At this stage, to complete the proof of 1), we simply need to observe that the inequality (A.11) holds for any two points x and y in $s_+^{-1}(]-\infty, +\infty[)$, not only when the coefficient (α, β) are positive rational number such that $\alpha + \beta = 1$, but more generally for any two positive real numbers (α, β) such $\alpha + \beta = 1$: as s_+ is non-increasing, this follows from a straightforward approximation argument. This establishes the concavity of s_+ .

The proof of 2) is similar, or follows from 1) applied to $-H$ instead of H . \square

A.3. Cramér's theorem. In this section, we assume that *the measure μ is a probability measure*. Observe that it implies that

$$\ell(0) = 0.$$

In particular, the interval $I := \ell^{-1}(\mathbb{R})$ is non-empty and contains 0. Moreover, the conditions $\mathbf{B}^>$ and $\mathbf{B}^<$ are satisfied.

A.3.1. The following theorem is the formulation, in measure theoretic language, of a general version of Cramer's theorem concerning the "empirical means"

$$\overline{X}_n := \frac{1}{n} \sum_{i=1}^n X_i$$

attached to a sequence $(X_i)_{i \geq 1}$ of independent and identically distributed real-valued random variables. We refer the reader to the article [CP11] of Cerf and Petit for a short and elegant proof.³⁵

³⁵The function s in *loc. cit.* corresponds to the function s_+ defined above. The assertions concerning s_- in Theorem A.3.1 follow from the ones concerning s_+ applied to the function $-H$ instead of H .

Theorem A.3.1. *For every $x \in \mathbb{R}$, we have:*

$$(A.14) \quad s_+(x) = \inf_{\xi \in \mathbb{R}_+} (\ell(\xi) - \xi x)$$

and

$$(A.15) \quad s_-(x) = \inf_{\xi \in \mathbb{R}_-} (\ell(\xi) - \xi x).$$

Moreover, for any $\xi \in \mathbb{R}_+$ (resp., for any $\xi \in \mathbb{R}_-$), we have:

$$(A.16) \quad \ell(\xi) = \sup_{x \in \mathbb{R}} (\xi x + s_+(x))$$

$$(A.17) \quad (\text{resp. } \ell(\xi) = \sup_{x \in \mathbb{R}} (\xi x + s_-(x))). \quad \square$$

Observe also that functions s_+ and s_- take their values in $[-\infty, 0]$. Moreover, according to (A.16) and (A.17), they are upper semi-continuous and satisfy:

$$(A.18) \quad \lim_{x \rightarrow -\infty} s_+(x) = 0,$$

$$\lim_{x \rightarrow (\sup_\mu H)_-} s_+(x) = s_+(\sup_\mu H) = \log \mu(H^{-1}(\sup_\mu H))$$

and

$$\lim_{x \rightarrow (\inf_\mu H)_+} s_-(x) = s_-(\inf_\mu H) = \log \mu(H^{-1}(\inf_\mu H)),$$

$$(A.19) \quad \lim_{x \rightarrow +\infty} s_-(x) = 0.$$

Under the terminology “Cramér’s Theorem”, one usually means the conjunction of the existence of the limits (A.12) and (A.13) for every $x \in \mathbb{R}$ that is established in Theorem A.2.5, together with the expressions (A.14) and (A.15) of these limits in terms of the log-Laplace transform ℓ stated in Theorem A.3.1.

The reader will also refer to [CP11] for additional references about large deviations and Cramér’s theorem, notably concerning the successive contributions, starting from Cramér’s seminal article ([Cra38]), which have led to the present simple and general formulation of Cramér’s theorem.

Let us also indicate that Chernoff’s version of Cramér’s theorem ([Che52], Theorem 1) would be enough to derive Theorem A.4.4 below.

A.3.2. The version of Cramér’s theorem formed by Theorems A.2.5 and A.3.1, when μ is a probability measure, may be supplemented by the following observations, intended to clarify its relation with other presentations of Cramér’s theory of large deviations (see for instance [Str11], Section 1.3).

Let us define $m_+ \in [-\infty, +\infty]$ and $m_- \in [-\infty, +\infty]$ as the “right and left derivatives”

$$m_+ := \inf_{\xi \in \mathbb{R}_+^*} \ell(\xi)/\xi = \lim_{\xi \rightarrow 0_+} \ell(\xi)/\xi =: \ell'_r(0).$$

and

$$m_- := \sup_{\xi \in \mathbb{R}_-^*} \ell(\xi)/\xi = \lim_{\xi \rightarrow 0_-} \ell(\xi)/\xi =: \ell'_l(0).$$

of the lower semi-continuous convex function ℓ at 0.

We clearly have:

$$m_+ \leq m_-$$

and, from Theorem A.3.1, we easily get:

Corollary A.3.2. *For any $x \in \mathbb{R}$, we have:*

$$(A.20) \quad s_+(x) = 0 \iff x \leq m_+$$

and

$$(A.21) \quad s_-(x) = 0 \iff x \geq m_-.$$

Moreover the function

$$(A.22) \quad s := \min(s_-, s_+) : \mathbb{R} \longrightarrow [-\infty, +\infty]$$

is upper semi-continuous and concave, and the functions ℓ and $-s$ may be deduced from each other by Legendre-Fenchel duality:

$$\text{for every } x \in \mathbb{R}, s(x) = \inf_{\xi \in \mathbb{R}} (\ell(\xi) - \xi x)$$

and

$$\text{for every } \xi \in \mathbb{R}, \ell(\xi) = \sup_{x \in \mathbb{R}} (\xi x + s(x)).$$

□

Observe finally that, if we denote by H^+ and H^- the positive and negative parts³⁶ of H , then the following conditions are equivalent:

- (1) $m_+ < +\infty$,
- (2) $I \cap \mathbb{R}_+^* \neq \emptyset$,
- (3) for some $\epsilon \in \mathbb{R}_+^*$, the function $\exp(\epsilon H_+)$ is μ -integrable,

and the following ones as well:

- (1) $m_- > -\infty$,
- (2) $I \cap \mathbb{R}_-^* \neq \emptyset$,
- (3) for some $\epsilon \in \mathbb{R}_+^*$, the function $\exp(\epsilon H_-)$ is μ -integrable.

In particular, if $m_+ < +\infty$ and $m_- > -\infty$, then I contains some neighborhood of 0 in \mathbb{R} , the function H is μ -integrable, and

$$m_+ = m_- = \ell'(0) = \int_E H d\mu.$$

More generally, if $m_+ < +\infty$ (resp., $m_- > -\infty$), the integral

$$\int_E H d\mu := \int_E H^+ d\mu - \int_E H^- d\mu$$

is well-defined in $[-\infty, +\infty[$ (resp., in $] -\infty, +\infty]$) and is easily seen to be equal to m_+ (resp., to m_-).

A.4. An extension of Cramér's Theorem concerning general positive measures. In this section, the measure μ is allowed to have a total mass $\mu(E)$ different of 1.

When $\mu(\mathcal{E})$ is finite, one may apply the results in the previous section with μ replaced by the probability measure

$$\mu_0 := \mu(\mathcal{E})^{-1} \mu.$$

In this way, one easily sees that Theorem A.3.1 still holds *ne varietur*, and that its consequences remain valid with minor modifications (in (A.18)-(A.19) and in (A.20)-(A.22), 0 has to be replaced by $\log \mu(\mathcal{E})$).

When the total mass $\mu(\mathcal{E})$ is $+\infty$, then for any $x \in \mathbb{R}$,

$$A_1^>(x) + A_1^<(x) = +\infty$$

³⁶namely the \mathbb{R}_+ -valued functions on \mathcal{E} defined by $H^+ := \max(H, 0)$ and $H^- := \max(-H, 0)$.

and therefore $s_+(x) = +\infty$ or $s_-(x) = +\infty$. Therefore the conditions $\mathbf{B}^>$ and $\mathbf{B}^<$ cannot be simultaneously satisfied, and only one of the two functions s_+ and s_- may have an interesting behavior.

In this section, we shall focus on s_- , and we will show that the results in Theorem A.3.1 concerning s_- admit a sensible generalization when H is bounded from below — say when $\inf_\mu H$ is non-negative.³⁷

A.4.1. In this paragraph, we assume that *the interval $I := \ell^{-1}(\mathbb{R})$ is not empty*, and we choose some element ξ_0 in I .

Then the integral

$$\int_{\mathcal{E}} e^{\xi_0 H} d\mu = e^{\ell(\xi_0)}$$

belongs to $]0, +\infty[$ and the measure

$$\mu_0 := \left(\int_{\mathcal{E}} e^{\xi_0 H} d\mu \right)^{-1} e^{\xi_0 H} \mu$$

is a probability measure on the measurable space $(\mathcal{E}, \mathcal{T})$. Therefore, we may apply the constructions and the results of the previous sections to the probability space $(\mathcal{E}, \mathcal{T}, \mu_0)$ and to the function H .

Let us denote by ℓ_0 , s_{0+} , and s_{0-} the functions ℓ , s_+ , and s_- associated to these new data. The following lemma is a straightforward consequence of the definitions:

Lemma A.4.1. *For every non-negative integer n , and every (ξ, x) in \mathbb{R}^2 , we have:*

$$(A.23) \quad \mu_0^{\otimes n} = e^{\xi_0 H_n - np(\xi_0)} \mu^{\otimes n},$$

$$(A.24) \quad \ell_0(\xi) = \ell(\xi + \xi_0) - \ell(\xi_0),$$

and

$$(A.25) \quad \frac{1}{n} \log \mu_0^{\otimes n}(H_n^{-1}(]-\infty, nx])) = \frac{1}{n} \log \int_{H_n^{-1}(]-\infty, nx]))} e^{\xi_0 H_n} d\mu^{\otimes n} - \ell(\xi_0).$$

□

By means of the relation (A.23)-(A.25), the assertions involving s_{0-} and p_0 in Theorems A.2.5 and A.3.1 applied to the probability space $(\mathcal{E}, \mathcal{T}, \mu_0)$ and to the function H may be reformulated as follows, without reference to the measures $\mu_0^{\otimes n}$:

Corollary A.4.2. *Let ξ_0 be a real number such that $\ell(\xi_0) < +\infty$.*

For every x in \mathbb{R} , we have:

$$(A.26) \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \log \int_{H_n^{-1}(]-\infty, nx]))} e^{\xi_0 H_n} d\mu^{\otimes n} = \inf_{\xi \in \mathbb{R}_-} (\ell(\xi + \xi_0) - \xi x).$$

Moreover, for every $\xi \in \mathbb{R}_-$,

$$(A.27) \quad \ell(\xi + \xi_0) = \sup_{x \in \mathbb{R}} \left(\xi x + \lim_{n \rightarrow +\infty} \frac{1}{n} \log \int_{H_n^{-1}(]-\infty, nx]))} e^{\xi_0 H_n} d\mu^{\otimes n} \right).$$

□

Observe that every term of sequence in the left-hand side of (A.26), and consequently its limit, belongs to $[-\infty, \ell(\xi_0)]$.

³⁷The interested reader will have no difficulty in extending the results of this section to the more general situation where $\inf_\mu H > -\infty$.

A.4.2. *An application.* In this paragraph, we apply the previous results to the situation where H assumes only non-negative values on \mathcal{E} .

We shall use the non-negativity of H through the following simple observation:

Lemma A.4.3. *Let us assume that $H(\mathcal{E}) \subset \mathbb{R}_+$, and let us consider a positive integer n and some elements x of \mathbb{R}_+ and η of \mathbb{R}_+^* .*

Then $H_n^{-1}(]-\infty, nx])$ is empty if $x < 0$ and equals $H_n^{-1}([0, nx])$ if $x \geq 0$. Therefore, every $P \in H_n^{-1}(]-\infty, nx])$ satisfies:

$$e^{-\eta nx} \leq e^{-\eta H_n(P)} \leq 1,$$

and consequently:

$$(A.28) \quad \frac{1}{n} \log A_n^<(x) - \eta x \leq \frac{1}{n} \log \int_{H_n^{-1}(]-\infty, nx])} e^{-\eta H_n} d\mu^{\otimes n} \leq \frac{1}{n} \log A_n^<(x).$$

□

We may now formulate the main result of this Appendix:

Theorem A.4.4. *Let us assume that the following conditions are satisfied:*

$$H(\mathcal{E}) \subset \mathbb{R}_+$$

and

$$(A.29) \quad \text{for any } \xi \in \mathbb{R}_-^*, \int_{\mathcal{E}} e^{\xi H} d\mu < +\infty.$$

Then, for every x in \mathbb{R} , we have:

$$(A.30) \quad s_-(x) = \inf_{\xi \in \mathbb{R}_-^*} (p(\xi) - \xi x).$$

In particular, Condition $\mathbf{B}^<$ is satisfied and the concave function $s_- : \mathbb{R} \rightarrow [-\infty, +\infty[$ is upper-semicontinuous.

Moreover, for every $\xi \in \mathbb{R}_-^*$, we have:

$$(A.31) \quad \ell(\xi) = \sup_{x \in \mathbb{R}} (\xi x + s_-(x)).$$

Proof. Let us consider x in \mathbb{R} and η in \mathbb{R}_+^* .

We may apply Corollary A.4.2 with $\xi_0 = -\eta$, and observe that, when n goes to $+\infty$, the inequalities (A.28) become

$$(A.32) \quad s_-(x) - \eta x \leq \lim_{n \rightarrow +\infty} \frac{1}{n} \log \int_{H_n^{-1}(]-\infty, nx])} e^{-\eta H_n} d\mu^{\otimes n} \leq s_-(x).$$

According to (A.26), the middle term in (A.32) is

$$\inf_{\xi \in \mathbb{R}_-} (\ell(\xi - \eta) - \xi x) = \inf_{\xi \in]-\infty, -\eta]} (\ell(\xi) - \xi x) - \eta x,$$

so that (A.32) may also be written:

$$s_-(x) - \eta x \leq \inf_{\xi \in]-\infty, -\eta]} (\ell(\xi) - \xi x) - \eta x \leq s_-(x).$$

By taking the limit of these inequalities when η goes to $0+$, we obtain (A.30).

Let ξ be an element of \mathbb{R}_-^* . The relation (A.27) for $\xi_0 := -\eta$, together with (A.32), shows that:

$$\sup_{x \in \mathbb{R}} (\xi x + s_-(x) - \eta x) \leq \ell(\xi - \eta) \leq \sup_{x \in \mathbb{R}} (\xi x + s_-(x)).$$

Since ℓ is continuous on \mathbb{R}_-^* , (A.31) follows. □

A.5. Reformulation and complements. In this section, for the convenience of the reader, we reformulate some of the results previously established in this Appendix without making reference to the formalism introduced in the previous sections.

To emphasize the possible thermodynamical interpretation of these results, we will introduce some new notation, related to the previous one through the following formulas:

$$\Psi(\beta) = \ell(-\beta) \quad \text{and} \quad S(x) = s_-(x).$$

A.5.1. *A scholium.* Let us recall that we consider a measure space $(\mathcal{E}, \mathcal{T}, \mu)$ defined by a set \mathcal{E} , a σ -algebra \mathcal{T} over \mathcal{E} , and a non-zero σ -finite non-negative measure $\mu : \mathcal{T} \rightarrow [0, +\infty]$. For every positive integer n , we also consider the product $\mu^{\otimes n}$ of n copies of the measure μ on \mathcal{E}^n equipped with the σ -algebra $\mathcal{T}^{\otimes n}$.

Besides, we consider a \mathcal{T} -measurable function

$$H : \mathcal{E} \rightarrow \mathbb{R}.$$

For every $\beta \in \mathbb{R}_+^*$ such that $e^{-\beta H}$ is μ -integrable, the integral $\int_{\mathcal{E}} e^{-\beta H} d\mu$ is a positive real number, and we let:

$$(A.33) \quad \Psi(\beta) := \log \int_{\mathcal{E}} e^{-\beta H} d\mu \quad (\in \mathbb{R}).$$

Under this integrability assumption, we also define:

$$F(\beta) := -\beta^{-1} \Psi(\beta).$$

Equivalently, $F(\beta)$ may be defined by the relation:

$$e^{-\beta F(\beta)} = \int_{\mathcal{E}} e^{-\beta H} d\mu.$$

The μ -integrability of the function $e^{-\beta H}$ for every $\beta \in \mathbb{R}_+^*$ is easily seen to be equivalent to the sub-exponential growth of the function

$$N : \mathbb{R}_+ \rightarrow [0, +\infty]$$

defined by

$$N(x) := \mu(H^{-1}[0, x]),$$

namely to the condition:

SE For every $x \in \mathbb{R}_+$, $N(x)$ is finite, and, when x goes to $+\infty$,

$$\log N(x) = o(x).$$

We shall also consider the essential infimum

$$\inf_{\mu} H := \inf\{x \in \mathbb{R}_+ \mid N(x) > 0\}$$

of H with respect to the measure space $(\mathcal{E}, \mathcal{T}, \mu)$.

Theorem A.5.1. *Let us keep the above notation and let us assume that Condition **SE** is satisfied and that $\mu(\mathcal{E}) = +\infty$.*

1) *For every x in $]\inf_{\mu} H, +\infty[$, the limit*

$$S(x) := \lim_{n \rightarrow +\infty} \frac{1}{n} \log \mu^{\otimes n}(\{(e_1, \dots, e_n) \in \mathcal{E}^n \mid H(e_1) + \dots + H(e_n) \leq nx\})$$

exists in \mathbb{R} .

The function $S :]\inf_{\mu} H, +\infty[\rightarrow \mathbb{R}$ is non-decreasing and concave, and satisfies

$$(A.34) \quad \lim_{x \rightarrow (\inf_{\mu} H)_+} S(x) = \mu(H^{-1}(\inf_{\mu} H)) \quad (\in [-\infty, +\infty]).$$

2) The function $\Psi : \mathbb{R}_+^* \rightarrow \mathbb{R}$ is real analytic and convex. Its derivative up to a sign

$$U := -\Psi'$$

satisfies, for every $\beta \in \mathbb{R}_+^*$,

$$(A.35) \quad U(\beta) := \frac{\int_{\mathcal{E}} H e^{-\beta H} d\mu}{\int_{\mathcal{E}} e^{-\beta H} d\mu}$$

and defines a decreasing real analytic diffeomorphism:

$$(A.36) \quad U : \mathbb{R}_+^* \xrightarrow{\sim}]\inf_{\mu} H, +\infty[.$$

3) The functions $-S(-.)$ and Ψ are Legendre-Fenchel transforms of each other.

Namely, for every $x \in]\inf_{\mu} H, +\infty[$,

$$(A.37) \quad S(x) = \inf_{\beta \in \mathbb{R}_+^*} (\beta x + \Psi(\beta)),$$

and, for every $\beta \in \mathbb{R}_+^*$,

$$(A.38) \quad \Psi(\beta) = \sup_{x \in]\inf_{\mu} H, +\infty[} (S(x) - \beta x).$$

Proof. Assertion 1) follows from Theorem A.4.4 and from Theorem A.2.5, 2).

Observe that H is not μ -almost everywhere constant — otherwise the conditions **SE** and $\mu(\mathcal{E}) = +\infty$ could not be both satisfied. Therefore $\Psi'' > 0$ on \mathbb{R}_+^* by Proposition A.1.1.

The expression (A.35) *à la Gibbs* for $U := -\Psi'$ is a straightforward consequence of Lebesgue integration theory. To complete the proof of 2), we are thus left to show that

$$(A.39) \quad \lim_{\beta \rightarrow 0_+} U(\beta) = +\infty$$

and

$$(A.40) \quad \lim_{\beta \rightarrow +\infty} U(\beta) = \inf_{\mu} H.$$

Since the function Ψ satisfies

$$\lim_{\beta \rightarrow 0_+} \Psi(\beta) = \int_{\mathcal{E}} d\mu = +\infty,$$

its derivative cannot stay bounded near zero. This proves (A.39).

To prove (A.40), simply observe that, according to (A.35), for any $\beta \in \mathbb{R}_+^*$,

$$U(\beta) \geq \inf_{\mu} H$$

and that, for any $\eta > \inf_{\mu} H$ and for any $\beta \in \mathbb{R}_+^*$,

$$\begin{aligned} \Psi(\beta) &\geq \log \int_{\mathcal{E}} e^{-\beta H} d\mu \\ &\geq \log \int_{H^{-1}([0, \eta])} e^{-\beta \eta} d\mu \\ &\geq \log N(\eta) - \beta \eta, \end{aligned}$$

so that $\lim_{\beta \rightarrow +\infty} \Psi'(\beta) \geq -\beta$.

Assertion 3) directly follows from Theorem A.4.4. □

The following corollary is now a consequence of the elementary theory of Legendre-Fenchel transforms of convex functions of one real variable³⁸:

Corollary A.5.2. *The function S is increasing and real analytic on $]\inf_\mu H, +\infty[$. Moreover its derivative establishes a decreasing real analytic diffeomorphism*

$$S' :]\inf_\mu H, +\infty[\xrightarrow{\sim} \mathbb{R}_+^*$$

inverse of the diffeomorphism (A.36).

For any $x \in]\inf_\mu H, +\infty[$, the infimum in the right-hand side of (A.37) is attained at a unique $\beta \in \mathbb{R}_+^$, namely*

$$(A.41) \quad \beta = S'(x).$$

Dually, for any $\beta \in \mathbb{R}_+^$, the supremum in the right-hand side of (A.38) is attained at a unique any $x \in]\inf_\mu H, +\infty[$, namely*

$$(A.42) \quad x = U(\beta).$$

Observe that, when $x \in]\inf_\mu H, +\infty[$ and $\beta \in \mathbb{R}_+^*$ satisfy the equivalent relations (A.41) and (A.42), then

$$(A.43) \quad S(x) = \beta x + \Psi(\beta),$$

or equivalently:

$$F(\beta) := -\beta^{-1}\Psi(\beta) = U(\beta) - \beta^{-1}S(x).$$

(“Expression of the free energy F in terms of the energy U , the temperature β^{-1} , and the entropy S ”).

Observe also that, according to (A.34), the following two conditions are equivalent:

$$\mu(\{e \in \mathcal{E} \mid H(e) = \inf_\mu H\}) = 1$$

and

$$\lim_{\beta \rightarrow +\infty} S(U^{-1}(\beta)) = 0$$

(“Nernst’s principle”).

A.5.2. Products and thermal equilibrium. The formalism summarized in the previous paragraph — that attaches functions Ψ and S to a measure space $(\mathcal{E}, \mathcal{T}, \mu)$ and to a non-negative function H on \mathcal{E} satisfying **SE** — satisfies a simple but remarkable, compatibility with finite products, that we want to discuss briefly.

Assume that, for any element i in some finite set I , we are given a measure space $(\mathcal{E}_i, \mathcal{T}_i, \mu_i)$ and a measurable function $H_i : \mathcal{E}_i \rightarrow \mathbb{R}_+$ as in the previous paragraph A.5.1 above.

Then we may form the product measure space $(\mathcal{E}, \mathcal{T}, \mu)$ defined by the set $\mathcal{E} := \prod_{i \in I} \mathcal{E}_i$ equipped with the σ -algebra $\mathcal{T} := \bigotimes_{i \in I} \mathcal{T}_i$ and the product measure $\mu := \bigotimes_{i \in I} \mu_i$.

We may also define a measurable function

$$H : \mathcal{E} \rightarrow \mathbb{R}_+$$

by the formula

$$H := \sum_{i \in I} \text{pr}_i^* H_i,$$

where $\text{pr}_i : \mathcal{E} \rightarrow \mathcal{E}_i$ denotes the projection on the i -th factor.

³⁸Basically it follows from the fact that the Legendre-Fenchel transform g of real analytic function f with positive second derivative is itself real analytic with positive second derivative, as it may be expressed by the classical Legendre duality relation : $g(\xi) + f(x) = x\xi$ where $\xi = f'(x)$ or, equivalently, $x = g'(\xi)$.

Let us assume that, for every $i \in I$, $(\mathcal{E}_i, \mathcal{T}_i, \mu_i)$ and H_i satisfy the condition **SE**, or equivalently that the functions $e^{-\beta H_i}$ is μ_i -integrable for every $\beta \in \mathbb{R}_+^*$.

Then $(\mathcal{E}, \mathcal{T}, \mu)$ and H are easily seen to satisfy **SE** also, as a consequence of Fubini's Theorem. Actually Fubini's Theorem shows that the function $\Psi : \mathbb{R}_+^* \rightarrow \mathbb{R}$ attached to the above data, defined as in A.5.1 by the formula

$$\Psi(\beta) := \log \int_{\mathcal{E}} e^{-\beta H} d\mu,$$

and the “partial functions” Ψ_i , $i \in I$, attached to each measure space $(\mathcal{E}_i, \mathcal{T}_i, \mu_i)$ equipped with the function H_i by the similar formula

$$\Psi_i(\beta) := \log \int_{\mathcal{E}_i} e^{-\beta H_i} d\mu_i,$$

satisfy the additivity relation:

$$(A.44) \quad \Psi = \sum_{i \in I} \Psi_i.$$

Let us also assume that $\mu_i(\mathcal{E}_i) = +\infty$ for every $i \in I$. Then we also have $\mu(\mathcal{E}) = +\infty$, and we may apply Theorem A.5.1 and Corollary A.5.2 to the data $(\mathcal{E}_i, \mathcal{T}_i, \mu_i, H_i)$, $i \in I$, and $(\mathcal{E}, \mathcal{T}, \mu, H)$.

Notably, we may define some concave functions

$$S_i :]\inf_{\mu_i} H_i, +\infty[\rightarrow \mathbb{R}, \quad \text{for } i \in I,$$

and

$$S :]\inf_{\mu} H, +\infty[\rightarrow \mathbb{R}.$$

Observe also that, as a straightforward consequence of the definitions, we have:

$$\inf_{\mu} H = \sum_{i \in I} \inf_{\mu_i} H_i.$$

The following proposition may be seen as a mathematical interpretation of the second law of thermodynamics:

Proposition A.5.3. 1) For each $i \in I$, let x_i be a real number in $]\inf_{\mu_i} H_i, +\infty[$.

Then the following inequality is satisfied:

$$(A.45) \quad \sum_{i \in I} S_i(x_i) \leq S\left(\sum_{i \in I} x_i\right).$$

Moreover equality holds in (A.45) if and only if the positive real numbers $S'(x_i)$, $i \in I$, are all equal. When this holds, if β denotes their common value, we also have:

$$\beta = S'\left(\sum_{i \in I} x_i\right).$$

2) Conversely, for any $x \in]\inf_{\mu} H, +\infty[$, there exists a unique family $(x_i)_{i \in I}$ in $\prod_{i \in I}]\inf_{\mu_i} H_i, +\infty[$ such that

$$x = \sum_{i \in I} x_i \quad \text{and} \quad S(x) = \sum_{i \in I} S_i(x_i).$$

Indeed, if $\beta = S'(x)$, it is given by

$$(x_i)_{i \in I} = (U_i(\beta))_{i \in I},$$

where $U_i = -\Psi'_i$.

Proof. Let $(x_i)_{i \in I}$ be an element of $\prod_{i \in I}]\inf_{\mu_i} H_i, +\infty[$. According to Corollary A.5.2, for every $i \in I$,

$$(A.46) \quad S(x_i) = \inf_{\beta > 0} (\beta x_i + \Psi_i(\beta)).$$

Moreover, the infimum is attained for a unique positive β , namely $S'(x_i)$.

Similarly, for $x := \sum_{i \in I} x_i$,

$$(A.47) \quad S(x) = \inf_{\beta > 0} (\beta x + \Psi(\beta)),$$

and the infimum is attained for a unique positive β , namely $S'(x)$.

Besides, the additivity relation (A.44) shows that, for every β in \mathbb{R}_+^* ,

$$\beta x + \Psi(\beta) = \sum_{i \in I} (\beta x_i + \Psi_i(\beta)).$$

Part 1) of the proposition directly follows from these observations. Part 2) follows from Part 1) and from the relation $\Psi' = \sum_{i \in I} \Psi'_i$. \square

A.5.3. An example: Gaussian integrals and Maxwell velocity distribution. In this paragraph, we discuss a simple but significant instance of the formalism summarized in paragraph A.5.1 and in Theorem A.5.1. This example may be seen as a mathematical counterpart of Maxwell's statistical approach to the theory of ideal gases. It is also included for comparison with the application in Section 4.4 of the above formalism to euclidean lattices — the present example appears as a “classical limit” of the discussion of Section 4.4.

Let V be a finite dimensional real vector space equipped with some euclidean norm $\|\cdot\|$.

We shall denote by λ the Lebesgue measure on V attached to this euclidean norm. It may be defined as the unique translation invariant Radon measure on V such that

$$\int_V e^{-\pi \|x\|^2} d\lambda(x) = 1.$$

(Compare 3.1.1 and equation (3.1).)

We may apply the formalism of this appendix to the measure space $(V, \mathcal{B}, \lambda)$, defined by V equipped with the Borel σ -algebra \mathcal{B} and with the Lebesgue measure λ , and to the function

$$H := (1/2m) \|\cdot\|^2$$

where m denotes some positive real number.

Then, for every β in \mathbb{R}_+^* , we have:

$$\int_V e^{-\beta \|p\|^2 / 2m} d\lambda(p) = (2\pi m / \beta)^{\dim V / 2}.$$

Therefore

$$(A.48) \quad \Psi(\beta) = (\dim V / 2) \log(2\pi m / \beta)$$

and

$$(A.49) \quad U(\beta) = -\Psi'(\beta) = \dim V / (2\beta).$$

The equations (A.41) and (A.42), that relates the “energy” x and the “inverse temperature” β , takes the form for any $x \in]\inf_{\lambda} H, +\infty[= \mathbb{R}_+^*$ and any $\beta \in \mathbb{R}_+^*$:

$$(A.50) \quad \beta x = \dim V / 2.$$

The function $S(x)$ may be computed directly from its definition.

Indeed, for any $x \in \mathbb{R}_+^*$ and any positive integer n , we have:

$$(A.51) \quad \lambda^{\otimes n} (\{(e_1, \dots, e_n) \in V^n \mid (1/2m)(\|e_1\|^2 + \dots + \|e_n\|^2) \leq nx\}) = v_{n \dim V} (2mnx)^{n(\dim V)/2}.$$

Here $v_{n \dim V}$ denotes the volume of the unit ball in the euclidean space of dimension $n \dim V$. It is given by:

$$(A.52) \quad v_{n \dim V} = \frac{\pi^{n(\dim V)/2}}{\Gamma(1 + n(\dim V)/2)}.$$

From (A.51) and (A.52), by a simple application of Stirling's formula, we get:

$$S(x) = \lim_{n \rightarrow +\infty} \log \left[v_{n \dim V} (2mnx)^{n(\dim V)/2} \right] = (\dim V/2)[1 + \log(4\pi mx / \dim V)].$$

In particular,

$$S'(x) = \dim V / (2x)$$

and we recover (A.50).

Conversely, combined with the expression (A.48) for the function Ψ , Part 3) of Theorem A.5.1 allows one to recover the asymptotic behaviour of the volume v_n of the n -dimensional unit ball, in the form:

$$v_n^{1/n} \sim \sqrt{2e\pi/n} \quad \text{when } n \rightarrow +\infty.$$

Finally, observe that when $m = (2\pi)^{-1}$ — the case relevant for the comparison with the application to euclidean lattices in Section 4.4 — the expressions for Ψ and S take the following simpler forms:

$$\Psi(\beta) = (\dim V/2) \log(1/\beta)$$

and

$$S(x) = (\dim V/2)[1 + \log(2x / \dim V)].$$

A.5.4. Relations with probability measures of maximal entropy. In this paragraph, we keep the notation recalled in paragraph A.5.1, and we assume that the hypotheses of Theorem A.5.1 are satisfied — namely we assume that the growth condition **SE** holds and that $\mu(\mathcal{E}) = +\infty$.

Let \mathcal{C} be the space of *probability measures on $(\mathcal{E}, \mathcal{T})$ absolutely continuous with respect to μ* .

By sending such a measure ν to its Radon-Nikodym derivative $f = d\mu/d\nu$, one establishes a bijection from \mathcal{C} to the convex subset

$$\{f \in L^1(\mathcal{E}, \mu) \mid f \geq 0 \text{ } \mu\text{-a.e. and } \int_{\mathcal{E}} f d\mu = 1\}$$

of $L^1(\mathcal{E}, \mu)$.

To any measure $\nu = f\mu$ in \mathcal{C} , we may attach its “energy”:

$$\epsilon(\nu) := \int_{\mathcal{E}} H d\nu = \int_{\mathcal{E}} H f d\mu \in [0, +\infty].$$

Lemma A.5.4. *Let $f : \mathcal{E} \rightarrow \mathbb{R}_+$ and $g : \mathcal{E} \rightarrow \mathbb{R}_+^*$ be two \mathcal{T} -measurable functions.*

1) *For every $x \in \mathcal{E}$,*

$$(A.53) \quad f(x) \log \frac{f(x)}{g(x)} \geq f(x) - g(x).$$

Moreover, equality holds in (A.53) if and only if $f(x) = g(x)$.

2) *If g is μ -integrable, then the negative part $(f \log(f/g))^-$ of $f \log(f/g)$ is μ -integrable, and therefore*

$$\int_{\mathcal{E}} f \log(f/g) d\mu$$

is well-defined in $] -\infty, +\infty]$.

3) *If f and g belongs to \mathcal{C} , then*

$$\int_{\mathcal{E}} f \log(f/g) d\mu$$

belongs to $[0, +\infty]$ and vanishes if and only if $f = g$ μ -almost everywhere.

Proof. For any $t \in \mathbb{R}_+$, we have:

$$t \log t \geq t - 1,$$

and equality holds if and only if $t = 1$. Applied to $t = f(x)/g(x)$, this implies 1).

From 1), assertions 2) and 3) immediately follow. \square

Proposition A.5.5. *Let $\nu = f\mu$ be an element of \mathcal{C} .*

1) *If $\epsilon(\nu) < +\infty$, then $(f \log f)^-$ is μ -integrable, and therefore the “information theoretic entropy” of ν with respect to μ*

$$I(\nu \mid \mu) := - \int_{\mathcal{E}} \log(d\mu/d\nu) d\nu = - \int_{\mathcal{E}} f \log f d\mu$$

is well defined in $[-\infty, +\infty[$.

2) *Let u and β be two positive real numbers such that $u = U(\beta)$.*

If $\epsilon(\nu) = u$, then

$$(A.54) \quad I(\nu \mid \mu) \leq S(u).$$

Moreover the equality is achieved in (A.54) for a unique measure ν of \mathcal{C} in $\epsilon^{-1}(u)$, namely for the measure

$$\nu_{\beta} := Z(\beta)^{-1} e^{-\beta H} \mu,$$

where

$$Z(\beta) := \int_{\mathcal{E}} e^{-\beta H} d\mu.$$

In substance, the content of Proposition A.5.5 goes back to the seminal work of Boltzmann and Gibbs on statistical mechanics³⁹. Similar results play also a central role in information theory and in statistics (see for instance [Kul97], notably Chapter 3). We refer the reader to [Geo03] (notably § 3.4) for additional informations and references.

Proof. Let ν be an element of \mathcal{C} such that $u := \epsilon(\nu)$ is finite, and let $\beta := U_{\overline{E}}^{-1}(u)$.

We may consider the measurable function

$$g_{\beta} := Z(\beta)^{-1} e^{-\beta H}.$$

It is everywhere positive on \mathcal{E} , and satisfies:

$$(A.55) \quad \int_{\mathcal{E}} g_{\beta} d\mu = 1$$

and

$$\log g_{\beta} = -\log Z(\beta) - \beta H = -\Psi(\beta) - \beta H.$$

In particular, we have:

$$(A.56) \quad \begin{aligned} f \log f &= f \log(f/g_{\beta}) + f \log g_{\beta} \\ &= f \log(f/g_{\beta}) - \Psi(\beta)f - \beta Hf. \end{aligned}$$

Also recall that

$$\int_{\mathcal{E}} f d\mu = 1$$

and

$$\int_{\mathcal{E}} Hf d\mu = u = U_{\overline{E}}(\beta) < +\infty.$$

³⁹See for instance Boltzmann’s memoirs [Bol72] and [Bol77], Chapter V. Our presentation is a straightforward generalization, in the framework of general measure theory, of the “axiomatic” approach of Gibbs in [Gib60], Chapter XI (Lemma A.5.4 above notably appears in *loc. cit.*, p. 130).

Consequently, using (A.43), we get:

$$(A.57) \quad \int_{\mathcal{E}} (\Psi(\beta)f + \beta Hf) d\mu = \Psi(\beta) + \beta u = S(u).$$

According to Lemma A.5.4, 2), the negative part $(f \log(f/g_\beta))^-$ of $f \log(f/g_\beta)$ is μ -integrable. Together with (A.56), this shows that the negative part $(f \log f)^-$ of $f \log f$ is μ -integrable. This establishes 1).

Moreover, according to Lemma A.5.4, 3), the integral $\int_{\mathcal{E}} f \log(f/g_\beta) d\mu$ is non-negative and vanishes if and only if $f = g_\beta$ μ -almost everywhere. Together with (A.56) and (A.57), this establishes 2). \square

APPENDIX B. NON-COMPLETE DISCRETE VALUATION RINGS AND CONTINUITY OF LINEAR FORMS ON PRODISCRETE MODULES

This Appendix is devoted to a discussion of the results of “automatic continuity” of Specker ([Spe50]) and Enochs ([Eno64]), in a setting adapted to their application to the categories CTC_A in Section 5.2.

B.1. Preliminary: maximal ideals, discrete valuation rings, and completions. Let R be a ring, and let \mathfrak{m} be a maximal ideal of R such that the local ring

$$R_{(\mathfrak{m})} := (R \setminus \mathfrak{m})^{-1}R$$

is a discrete valuation ring. Its maximal ideal is $\mathfrak{m}_{(\mathfrak{m})} := (R \setminus \mathfrak{m})^{-1}\mathfrak{m}$. We shall denote by

$$\iota : R \longrightarrow R_{(\mathfrak{m})}$$

the canonical morphism of rings, which send an element x of R to $\iota(x) := x/1$.

We may consider the \mathfrak{m} -adic completion of R at \mathfrak{m} :

$$\hat{R}_{\mathfrak{m}} := \varprojlim_n R/\mathfrak{m}^n.$$

It is a local ring, of maximal ideal $\hat{\mathfrak{m}} := \varprojlim_n \mathfrak{m}/\mathfrak{m}^n$.

As \mathfrak{m} is a maximal ideal, the localization $R_{(\mathfrak{m})}$ may be identified with a subring of $\hat{R}_{\mathfrak{m}}$, so that the canonical morphism from R to $\hat{R}_{\mathfrak{m}}$ becomes the composition $R \xrightarrow{\iota} R_{(\mathfrak{m})} \hookrightarrow \hat{R}_{\mathfrak{m}}$. Moreover the \mathfrak{m} -adic (resp. $\mathfrak{m}_{(\mathfrak{m})}$ -adic, resp. $\hat{\mathfrak{m}}$ -adic) filtrations on R (resp. $R_{(\mathfrak{m})}$, resp. $\hat{R}_{\mathfrak{m}}$) are strictly compatible.

In particular, $\hat{R}_{\mathfrak{m}}$ may be identified with the $\mathfrak{m}_{(\mathfrak{m})}$ -adic completion of $R_{(\mathfrak{m})}$, and therefore is a complete discrete valuation ring.

We shall denote by v the \mathfrak{m} -adic⁴⁰ valuation on $R_{(\mathfrak{m})}$ and $\hat{R}_{\mathfrak{m}}$ and by $|\cdot| = e^{-v}$ the associated absolute value. For simplicity, we shall also denote by v and $|\cdot|$ their composition with the morphism $\iota : R \longrightarrow \hat{R}_{\mathfrak{m}}$.

Let us consider:

$$\mathcal{L} = \{(\lambda_i)_{i \in \mathbb{N}} \in \hat{R}_{\mathfrak{m}}^{\mathbb{N}} \mid \lim_{i \rightarrow +\infty} |\lambda_i| = 0\}.$$

For any $\lambda = (\lambda_i)_{i \in \mathbb{N}}$ in \mathcal{L} , we may define a R -linear map

$$\Sigma_{\lambda} : R^{\mathbb{N}} \longrightarrow \hat{R}_{\mathfrak{m}}$$

by letting:

$$\Sigma_{\lambda}((x_i)_{i \in \mathbb{N}}) := \sum_{i \in \mathbb{N}} \lambda_i \iota(x_i).$$

⁴⁰or, more correctly $\mathfrak{m}_{(\mathfrak{m})}$ - or $\hat{\mathfrak{m}}$ -adic.

Proposition B.1.1. *Let $\lambda := (\lambda_i)_{i \in \mathbb{N}}$ be an element of \mathcal{L} such that $I := \{i \in \mathbb{N} \mid \lambda_i \neq 0\}$ is infinite. If we define*

$$n := \min_{i \in \mathbb{N}} v(\lambda_i),$$

then we have:

$$(B.1) \quad \Sigma_\lambda(R^\mathbb{N}) = \hat{\mathfrak{m}}^n.$$

Proof of Proposition B.1.1. Let us choose a bijection $\psi : \mathbb{N} \xrightarrow{\sim} I$. The integer n and the image of Σ_λ are clearly unchanged if we replace the sequence $\lambda := (\lambda_i)_{i \in \mathbb{N}}$ by $(\lambda_{\psi(i)})_{i \in \mathbb{N}}$. Therefore, to establish Proposition B.1.1, we may assume that λ belongs to $(R \setminus \{0\})^\mathbb{N}$ and that the sequence of non-negative integers $n_i := v(\lambda_i)$ ($i \in \mathbb{N}$) is such that $n := \min_{i \in \mathbb{N}} n_i = n_0$.

For any $(x_i)_{i \in \mathbb{N}}$ in $R^\mathbb{N}$ and any i in \mathbb{N} , the product $\lambda_i \iota(x_i)$ belongs to $\hat{\mathfrak{m}}^{n_i}$, and *a fortiori* to $\hat{\mathfrak{m}}^n$. This implies that $\Sigma_\lambda(R^\mathbb{N})$ is contained in $\hat{\mathfrak{m}}^n$.

Conversely, let α be an element of $\hat{\mathfrak{m}}^n$. Observe that, for any $k \in \mathbb{N}$, $\lambda_k \iota(R)$ is dense in $\lambda_k \hat{R}_\mathfrak{m} = \hat{\mathfrak{m}}^{n_k}$. Therefore we may inductively construct a sequence $(x_i)_{i \in \mathbb{N}}$ such that, for any $k \in \mathbb{N}$,

$$v(\alpha - \sum_{i=0}^k \lambda_i \iota(x_i)) \geq n_{k+1}.$$

Then

$$\alpha = \sum_{i \in \mathbb{N}} \lambda_i \iota(x_i) = \Sigma_\lambda((x_i)_{i \in \mathbb{N}}).$$

□

Observe that, for any natural integer k , by applying Proposition B.1.1 to the sequence $(\lambda_{i+k})_{i \in \mathbb{N}}$, that still belongs to \mathcal{L} , we obtain:

Corollary B.1.2. *Under the assumption of Proposition B.1.1, the map Σ_λ is continuous and open from $R^\mathbb{N}$, equipped with the product topology of the discrete topology on each of the factors R , onto $\hat{\mathfrak{m}}^n$ equipped with the $\hat{\mathfrak{m}}$ -adic topology.*

Actually, for any $k \in \mathbb{N}$, if we define $n_k := \min_{i \in \mathbb{N}_{\geq k}} v(\lambda_i)$, then

$$\Sigma_\lambda(\{0\}^k \oplus R^{\mathbb{N}_{\geq k}}) = \hat{\mathfrak{m}}^{n_k},$$

while $\lim_{k \rightarrow +\infty} n_k = +\infty$.

□

B.2. Continuity of linear forms on prodiscrete modules. We keep the notation of the previous section, and we consider a topological module M over the ring R equipped with the discrete topology.

The topological R -module M is complete and prodiscrete, with a countable basis of neighborhoods of 0 precisely if it is isomorphic (as a topological module) to the projective limit $\varprojlim_k M_k$ of some projective system

$$M_0 \xleftarrow{q_0} M_1 \xleftarrow{q_1} M_2 \xleftarrow{q_2} \dots$$

of discrete R -modules. (The maps q_i may actually be assumed surjective.)

If we denote by

$$p_i : M \longrightarrow M_i$$

the canonical projection maps, a sequence $(m_i)_{i \in \mathbb{N}} \in M^\mathbb{N}$ converges to zero in M if and only if, for every $k \in \mathbb{N}$, the projection $p_k(m_i)$ vanishes for i large enough (depending on k).

For any such sequence $(m_i)_{i \in \mathbb{N}}$ in $M^\mathbb{N}$ and for any sequence $(r_i)_{i \in \mathbb{N}}$, the series $\sum_{i \in \mathbb{N}} r_i m_i$ converges in M .

Proposition B.2.1. *Let M be a complete prodiscrete topological R -module, with a countable basis of neighborhoods of 0, and let $\phi : M \rightarrow \hat{R}_{\mathfrak{m}}$ be a R -linear map.*

If ϕ is not continuous when M is equipped with its prodiscrete topology and $\hat{R}_{\mathfrak{m}}$ with the discrete topology, then there exists $k \in \mathbb{N}$ such that $\phi(M) = \hat{\mathfrak{m}}^k$.

Proof. Let us denote by π an element of $\mathfrak{m} \setminus \mathfrak{m}^2$. (The set $\mathfrak{m} \setminus \mathfrak{m}^2$ is indeed not empty, since $\mathfrak{m}_{(\mathfrak{m})} \neq \mathfrak{m}_{(\mathfrak{m})}^2$ and therefore $\mathfrak{m} \neq \mathfrak{m}^2$.)

If ϕ is not continuous, then there exists a sequence $(m_i)_{i \in \mathbb{N}}$ in $M^{\mathbb{N}}$ which converges to 0 in M and such that $(\phi(m_i))_{i \in \mathbb{N}}$, and therefore $(\lambda_i)_{i \in \mathbb{N}} := (\iota(\pi)^i \phi(m_i))_{i \in \mathbb{N}}$, belongs to $\hat{R}_{\mathfrak{m}}^{(\mathbb{N})}$.

The family $(\lambda) = (\lambda_i)_{i \in \mathbb{N}}$ satisfies the assumptions of Proposition B.1.1, and therefore, for some non-negative integer n , $\Sigma_{\lambda}(R^{\mathbb{N}}) = \hat{\mathfrak{m}}^n$.

We are going to prove that any element of $\Sigma_{\lambda}(R^{\mathbb{N}})$ belongs to the image of ϕ . This will show that $\phi(M)$ contains $\hat{\mathfrak{m}}^n$. Since $\phi(M)$ is a R -submodule of $\hat{R}_{\mathfrak{m}}$, this will complete the proof.

To achieve, let $(x_i)_{i \in \mathbb{N}}$ be any sequence in $R^{\mathbb{N}}$. We may form the following sum, convergent in M :

$$m := \sum_{i \in \mathbb{N}} \pi^i x_i m_i.$$

For any non-negative integer k , we have:

$$(B.2) \quad m = \sum_{0 \leq i \leq k} \pi^i x_i m_i + \pi^{k+1} r_k,$$

where r_k is defined as the convergent sum in M :

$$r_k := \sum_{i \in \mathbb{N}} \pi^i x_{i+m+1} m_{i+m+1}.$$

By applying ϕ to the relation (B.2), we see that, for every $k \in \mathbb{N}$,

$$\phi(m) - \sum_{0 \leq i \leq k} \iota(\pi^i x_i) \phi(m_i) \in \mathfrak{m}^{k+1}.$$

Finally,

$$\Sigma_{\lambda}((x_i)_{i \in \mathbb{N}}) = \sum_{i \in \mathbb{N}} \iota(\pi)^i \phi(m_i) \iota(x_i) = \phi(m).$$

□

Observe that, when R is a domain, the morphism $\iota : R \rightarrow R_{(\mathfrak{m})}$ is injective and $\iota(R)$ contains $\hat{\mathfrak{m}}^k$ for some $k \in \mathbb{N}$ if and only if R is \mathfrak{m} -adically complete. Therefore, from Proposition B.2.1, we immediately derive:

Corollary B.2.2. *Let M be a complete prodiscrete topological R -module, with a countable basis of neighborhoods of 0.*

If R is a domain and is not \mathfrak{m} -adically complete, then any R -linear map $\phi : M \rightarrow R$ is continuous when M is equipped with its prodiscrete topology and R with the discrete topology. □

APPENDIX C. MEASURES ON COUNTABLE SETS AND THEIR PROJECTIVE LIMITS

This Appendix is devoted to various results concerning measure theory on the Polish spaces defined as projective limits of countable systems of countable discrete sets that are used in the proofs of Section 8.

C.1. Finite measures on countable sets. Let D be a (possibly finite) countable set. The real vector space $\mathcal{M}^b(D)$ of real bounded measures on D (equipped with the σ -algebra $\mathcal{P}(D)$ of all subsets of D) may be identified with the vector space $l^1(D)$:

$$(C.1) \quad \begin{aligned} \mathcal{M}^b(D) &\xrightarrow{\sim} l^1(D) \\ \mu &\longmapsto (\mu(\{x\}))_{x \in D}. \end{aligned}$$

This isomorphism maps the cone $\mathcal{M}_+^b(D)$ of positive bounded measures onto

$$l_+^1(D) = l^1(D) \cap \mathbb{R}_+^D.$$

Moreover the total mass of some measure $\mu \in \mathcal{M}^b(D)$ coincides with the l^1 -norm of its image by the isomorphism (C.1):

$$\|\mu\| = \sum_{x \in D} |\mu(\{x\})|.$$

On the real vector space $\mathcal{M}^b(D)$, or equivalently on $l^1(D)$, we may consider the following separated locally convex topologies:

- (i) the topology of vague convergence of measures in $\mathcal{M}^b(D)$, or equivalently the topology on $l^1(D)$ induced by the topology of pointwise convergence on \mathbb{R}^D , or the $\sigma(l^1(D), \mathbb{R}^{(D)})$ -topology on $l^1(D)$;
- (ii) the topology of narrow convergence of measures in $\mathcal{M}^b(D)$, that is the $\sigma(l^1(D), l^\infty(D))$ -topology on $l^1(D)$;
- (iii) the topology defined by the “total mass norm” on $\mathcal{M}^b(D)$ (which coincides with the l^1 -norm on $l^1(D)$).

The first of these topologies is strictly finer than the second one, and the second one strictly finer than the third one. The following proposition compares the induced topologies on the cone $\mathcal{M}_+^b(D)$, and notably asserts that the topology of narrow convergence and the topology of norm convergence on $\mathcal{M}_+^b(D)$ coincide:

Proposition C.1.1. *Let (μ_i) be a sequence, or more generally a net, of elements of $\mathcal{M}_+^b(D)$ which converges vaguely to an element μ of $\mathcal{M}_+^b(D)$. Then the following conditions are equivalent:*

- (1) (μ_i) converges to μ in the topology of narrow convergence on $\mathcal{M}^b(D)$;
- (2) $\lim_i \mu_i(D) = \mu(D)$;
- (3) the total mass $\mu_i(D)$ stays bounded (for i large enough); moreover, for every $\epsilon \in \mathbb{R}_+^*$, there exists a finite subset F of D such that, for i large enough, $\mu_i(D \setminus F) < \epsilon$;
- (4) $\lim_i \|\mu_i - \mu\| = 0$.

This is well-known, and the implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$ are indeed easily established.

In this article, we shall say that a sequence (μ_i) in $\mathcal{M}_+^b(D)$ converges to some measure $\mu \in \mathcal{M}_+^b(D)$ when the above equivalent conditions are satisfied.

The following convergence criterion plays a central role in our study of the θ -invariants of infinite dimensional hermitian vector bundles.

Proposition C.1.2. *Let $(\mu_i)_{i \in \mathbb{N}}$ be a sequence in $\mathcal{M}_+^b(D)$. If there exists $(t_i)_{i \in \mathbb{N}}$ in $l^1(\mathbb{N})$ such that, for every $i \in \mathbb{N}$,*

$$(C.2) \quad \mu_{i+1} \leq e^{t_i} \mu_i,$$

then the sequence $(\mu_i)_{i \in \mathbb{N}}$ converges to some $\mu \in \mathcal{M}_+^b(D)$.

Proof. When the condition (C.2) is satisfied by $t_i = 0$ for every $i \in \mathbb{N}$ — that is, when $\mu_{i+1} \leq \mu_i$ for every $i \in \mathbb{N}$ — then, to any subset X of D , we may associate the limit

$$\mu(X) := \lim_{i \rightarrow +\infty} \mu_i(X)$$

of the non-increasing sequence $(\mu_i(X))_{i \in \mathbb{N}}$ in \mathbb{R}_+ . It is straightforward that this defines an element μ of $\mathcal{M}_+^b(D)$ and that $(\mu_i)_{i \in \mathbb{N}}$ converges to μ .

One reduces the general case of the Proposition to this special case by letting, for every $i \in \mathbb{N}$,

$$\nu_i := e^{-\sum_{0 \leq j < i} t_j} \mu_i.$$

Indeed, $(\nu_i)_{i \in \mathbb{N}}$ is then a sequence in $\mathcal{M}_+^b(D)$ such that $\nu_{i+1} \leq \nu_i$ for every $i \in \mathbb{N}$, and therefore admits a limit ν in $\mathcal{M}_+^b(D)$. Consequently, $(\mu_i)_{i \in \mathbb{N}}$ converges to

$$\mu := e^{\sum_{j \in \mathbb{N}} t_j} \nu. \quad \square$$

Recall that the construction which associates, to some countable set D , the cone $\mathcal{M}_+^b(D)$ equipped with the topology of narrow convergence (or equivalently, of norm convergence) is functorial.

Namely, for any map $f : D \rightarrow D'$ between two countable sets D and D' , we may define a \mathbb{R} -linear map

$$f_* : \mathcal{M}^b(D) \rightarrow \mathcal{M}^b(D')$$

by letting, for any $\mu \in \mathcal{M}^b(D)$ and any $X' \subset D'$,

$$f_* \mu(X') := \mu(f^{-1}(X')).$$

The map f_* maps $\mathcal{M}_+^b(D)$ to $\mathcal{M}_+^b(D')$ and is continuous when $\mathcal{M}^b(D)$ and $\mathcal{M}^b(D')$ are equipped both with the topology of narrow convergence or of norm convergence. Moreover, if $f' : D' \rightarrow D''$ is a second map, with range some countable set D'' , we have:

$$(f' \circ f)_* = f'_* \circ f_*.$$

C.2. Finite measures on projective limits of countable sets. Consider a projective system

$$(C.3) \quad D_\bullet : D_0 \xleftarrow{q_0} D_1 \xleftarrow{q_1} \dots \xleftarrow{q_{i-1}} D_i \xleftarrow{q_i} D_{i+1} \xleftarrow{q_{i+1}} \dots$$

of countable sets. We may equip each of them with the discrete topology, and the projective limit of this system

$$\widehat{D} := \varprojlim_i D_i$$

with the associated projective limit topology. It is a Polish space, and we shall denote by $\mathcal{M}^b(\widehat{D})$ the real vector space of real bounded Borel measures on \widehat{D} and by $\mathcal{M}_+^b(\widehat{D})$ the cone in $\mathcal{M}^b(\widehat{D})$ of positive bounded Borel measures on \widehat{D} .

Recall that every measure in $\mathcal{M}_+^b(\widehat{D})$ is regular and tight. Moreover the cone $\mathcal{M}_+^b(\widehat{D})$ may be identified with the projective limit of the projective system

$$\mathcal{M}_+^b(D_\bullet) : \mathcal{M}_+^b(D_0) \xleftarrow{q_{0*}} \mathcal{M}_+^b(D_1) \xleftarrow{q_{1*}} \dots \xleftarrow{q_{i-1,*}} \mathcal{M}_+^b(D_i) \xleftarrow{q_{i*}} \mathcal{M}_+^b(D_{i+1}) \xleftarrow{q_{i+1,*}} \dots$$

deduced from (C.3) by application of the functor \mathcal{M}_+^b . Indeed, if

$$p_i : \widehat{D} \rightarrow D_i$$

denotes, for any $i \in \mathbb{N}$, the canonical map from \widehat{D} to D_i , then the map which sends a measure in $\mathcal{M}_+^b(\widehat{D})$ to the family of its direct images by the p_i 's defines a bijection

$$(C.4) \quad \begin{array}{ccc} \mathcal{M}_+^b(\widehat{D}) & \xrightarrow{\sim} & \varprojlim_i \mathcal{M}_+^b(D_i) (\subset \prod_{i \in \mathbb{N}} \mathcal{M}_+^b(D_i)) \\ \mu & \longmapsto & (p_{i*} \mu)_{i \in \mathbb{N}}, \end{array}$$

according to a classical theorem of Kolmogorov ([Kol33], Section III.4; see also [Bou69], Section 4.3).

For every $(i, j) \in \mathbb{N}^2$ such that $i \leq j$, we let:

$$p_{ij} := q_i \circ q_{i+1} \circ \cdots \circ q_{j-1} : D_j \longrightarrow D_i.$$

Proposition C.2.1. *Let $(\gamma_i)_{i \in \mathbb{N}}$ be an element of $\prod_{i \in \mathbb{N}} \mathcal{M}_+^b(D_i)$ and let $(\lambda_i)_{i \in \mathbb{N}} \in l^1(\mathbb{N})$ such that, for every $j \in \mathbb{N}$,*

$$(C.5) \quad q_{j*} \gamma_{j+1} \leq e^{\lambda_j} \gamma_j.$$

Then, for every $i \in \mathbb{N}$, the sequence $(p_{ij} \gamma_j)_{j \in \mathbb{N}_{\geq i}}$ converges to some limit μ_i in $\mathcal{M}_+^b(D_i)$. Moreover there exists a unique measure $\mu \in \mathcal{M}_+^b(\widehat{D})$ such that, for any $i \in \mathbb{N}$,*

$$\mu_i = p_{i*} \mu.$$

In particular, the sequence

$$(\gamma_j(D_j))_{j \in \mathbb{N}} = (p_{0,j*} \gamma_j(D_0))_{j \in \mathbb{N}}$$

converges in \mathbb{R}_+ and, for any $i \in \mathbb{N}$,

$$(C.6) \quad \mu(\widehat{D}) = \mu_i(D_i) = \lim_{j \rightarrow +\infty} \gamma_j(D_j).$$

Proof. From (C.5), by application of p_{ij*} , we derive that, for any $(i, j) \in \mathbb{N}^2$ such that $i \leq j$,

$$p_{i,j+1*} \gamma_{j+1} \leq e^{\lambda_j} p_{ij*} \gamma_j.$$

The convergence of the sequence $(p_{ij*} \gamma_j)_{j \in \mathbb{N}_{\geq i}}$ to some limit μ_i in $\mathcal{M}_+^b(D_i)$ therefore follows from Proposition C.1.2. Moreover the continuity of the maps q_{i*} with respect to the norm topology shows that, for every $i \in \mathbb{N}$,

$$\mu_i = q_{i*} \mu_{i+1}.$$

The existence and the unicity of μ then follows from Kolmogorov's theorem (C.4). \square

The following Proposition provides an alternative interpretation of the convergence condition on the sequence $(\gamma_i)_{i \in \mathbb{N}}$ which appears in Proposition C.2.1. (We refer the reader to [Sch73], Appendix, or to [Bil99], Chapter 1, for basic results concerning the topology of narrow convergence — also called topology of weak convergence in [Bil99] — on the space of bounded measures on the Polish space \widehat{D} .)

Proposition C.2.2. *Let $(\gamma_i)_{i \in \mathbb{N}}$ be a family of measures in $\prod_{i \in \mathbb{N}} \mathcal{M}_+^b(D_i)$ and, for every $i \in \mathbb{N}$, let $\tilde{\gamma}_i$ be a measure in $\mathcal{M}_+^b(\widehat{D})$ such that*

$$(C.7) \quad p_{i*} \tilde{\gamma}_i = \gamma_i.$$

Then the following two conditions are equivalent:

Conv₁ : *For every $i \in \mathbb{N}$, the sequence $(p_{ij*} \gamma_j)_{j \in \mathbb{N}_{\geq i}}$ converges to some measure μ_i in $\mathcal{M}_+^b(D_i)$.*

Conv₂ : *The sequence $(\tilde{\gamma}_j)_{j \in \mathbb{N}}$ convergence to some measure μ in the topology of narrow convergence on $\mathcal{M}_+^b(\widehat{D})$.*

When these conditions hold, the measure μ is the unique element of $\mathcal{M}_+^b(\widehat{D})$ such that $p_{i} \mu = \mu_i$ for any $i \in \mathbb{N}$.*

When the conditions **Conv₁** and **Conv₂** of Proposition C.2.2 are satisfied, we shall say that *the sequence $(\gamma_i)_{i \in \mathbb{N}}$ satisfies condition **Conv*** and that μ is the *limit measure* associated to $(\gamma_i)_{i \in \mathbb{N}}$.

Observe that, when the maps $q_i : D_{i+1} \rightarrow D_i$, or equivalently the maps $p_i : \widehat{D} \rightarrow D_i$, are all surjective then for any sequence $(\gamma_i)_{i \in \mathbb{N}}$ in $\prod_{i \in \mathbb{N}} \mathcal{M}_+^b(D_i)$ as above, there exists a sequence $(\tilde{\gamma}_i)_{i \in \mathbb{N}}$ in $\mathcal{M}_+^b(\widehat{D})^{\mathbb{N}}$ such that the conditions (C.7) hold. (Indeed, if for any $x \in D_i$, we denote by \tilde{x} a point in $p_i^{-1}(x)$, we may simply define $\tilde{\gamma}_i$ as $\sum_{x \in D_i} \gamma_i(x) \delta_{\tilde{x}}$.)

Proof. Observe that, for any $(i, j) \in \mathbb{N}^2$ such that $i \leq j$, we have

$$p_{ij*}\gamma_j = p_{ij*}p_{j*}\tilde{\gamma}_j = p_{i*}\tilde{\gamma}_j.$$

The continuity of the maps $p_{i*} : \mathcal{M}^b(\hat{D}) \rightarrow \mathcal{M}^b(D_i)$ (for the topology of narrow convergence) therefore establishes the implication **Conv**₁ \implies **Conv**₂, with $\mu_i = p_{i*}\mu$ for every $i \in \mathbb{N}$. The unicity of the measure μ satisfying these conditions follows from the injectivity assertion in Kolmogorov's theorem (C.4).

Conversely, let us assume that **Conv**₁ is satisfied. The continuity of the maps $p_{ii'*} : \mathcal{M}^b(D_{i'}) \rightarrow \mathcal{M}^b(D_i)$, ($0 \leq i \leq i'$) in the topology of narrow convergence shows that the measures μ_i satisfy the conditions

$$p_{ii'*}\mu_{i'} = \mu_i.$$

According to Kolmogorov's theorem (C.4), there exists a (unique) measure $\mu \in \mathcal{M}_+^b(\hat{D})$ such that $\mu_i = p_{i*}\mu$ for every $i \in \mathbb{N}$.

To complete the proof, we are left to show that $(\tilde{\gamma}_i)_{i \in \mathbb{N}}$ converges to μ in the topology of narrow convergence. Recall that this means that, for every function f in the space $C^b(\hat{D}, \mathbb{R})$ of bounded continuous real valued functions on \hat{D} , we have:

$$(C.8) \quad \lim_{j \rightarrow +\infty} \int_{\hat{D}} f d\tilde{\gamma}_j = \int_{\hat{D}} f d\mu.$$

Also recall that, if d denotes a metric on \hat{D} that defines its topology, this condition is satisfied as soon as it is satisfied by any f in the subspace $C_u^b(\hat{D}, \mathbb{R})$ of bounded real valued functions on \hat{D} which are *uniformly* continuous with respect to d (see for instance [Bil99], Theorem 2.1⁴¹).

In turn, condition (C.8) is satisfied for every f in $C_u^b(\hat{D}, \mathbb{R})$ if it is satisfied for every f in some subset \mathcal{E} of $C_u^b(\hat{D}, \mathbb{R})$ dense in the topology of uniform convergence.

To prove that this last condition is indeed fulfilled, choose a decreasing sequence $(\epsilon_i)_{i \in \mathbb{N}}$ in $\mathbb{R}_+^{\mathbb{N}}$ such that $\lim_{i \rightarrow +\infty} \epsilon_i = 0$ and equip \hat{D} with a metric d such that, for any $x \in \hat{D}$ and any $i \in \mathbb{N}$:

$$\{x' \in \hat{D} \mid d(x', x) \leq \epsilon_i\} = p_i^{-1}(p_i(x)).$$

(For instance, we may choose ϵ_{-1} in $] \epsilon_0, +\infty[$ and consider the metric d defined by

$$d(x, y) := \epsilon_{k-1}, \text{ where } k := \min\{i \in \mathbb{N} \mid p_i(x) \neq p_i(y)\}$$

for any $(x, y) \in \hat{D}^2$ such that $x \neq y$.) Any such metric defines the topology of \hat{D} . Moreover, for every $i \in \mathbb{N}$, the image of

$$p_i^* := . \circ p_i : l^\infty(D_i) \longrightarrow C^b(\hat{D}, \mathbb{R})$$

belongs to the subspace $C_u^b(\hat{D}, \mathbb{R})$ of functions uniformly continuous with respect to d , and the union

$$\mathcal{E} := \bigcup_{i \in \mathbb{N}} p_i^*(l^\infty(D_i))$$

is dense in $C_u^b(\hat{D}, \mathbb{R})$ equipped with the topology of uniform convergence.

Finally, if f is an element of \mathcal{E} , of the form $p_i^*\phi$ for some $i \in \mathbb{N}$ and some $\phi \in l^\infty(D_i)$, then, for any $j \in \mathbb{N}_{\geq i}$,

$$\int_{\hat{D}} f d\tilde{\gamma}_j = \int_{\hat{D}} p_i^*\phi d\tilde{\gamma}_j = \int_{D_i} \phi dp_{i*}\tilde{\gamma}_j = \int_{D_i} \phi dp_{ij*}\gamma_j.$$

When j goes to $+\infty$, this converges to

$$\int_{D_i} \phi d\mu_i = \int_{\hat{D}} p_i^*\phi d\mu = \int_{\hat{D}} f d\mu.$$

⁴¹In *loc. cit.*, only the narrow (a.k.a. weak) convergence of *probability* measures is considered. The case of *bounded positive* measures easily reduces to this one.

Consequently, condition (C.8) is satisfied by f . \square

Proposition C.2.3. *Let $(\gamma_i)_{i \in \mathbb{N}}$ be a family of measures in $\prod_{i \in \mathbb{N}} \mathcal{M}_+^b(D_i)$.*

If the sequence $(\gamma_i(D_i))_{i \in \mathbb{N}}$ converges in \mathbb{R}_+ and if \mathcal{C} is a countable subset of \widehat{D} such that, for any $x \in \mathcal{C}$, the sequence $(\gamma_j(\{p_j(x)\}))_{j \in \mathbb{N}}$ admits a limit $\gamma(x)$ in \mathbb{R}_+ and if

$$(C.9) \quad \sum_{x \in \mathcal{C}} \gamma(x) = \lim_{i \rightarrow +\infty} \gamma_i(D_i),$$

*then the sequence $(\gamma_i)_{i \in \mathbb{N}}$ satisfies condition **Conv** and the associated limit measure is*

$$\mu := \sum_{x \in \mathcal{C}} \gamma(x) \delta_x.$$

Proof. This will follow from the following

Lemma C.2.4. *For every $\epsilon \in \mathbb{R}_+^*$, there exists $i(\epsilon) \in \mathbb{N}$ and a finite subset $F_\epsilon \subset \mathcal{C}$ such that*

$$(C.10) \quad \mu(\mathcal{C} \setminus F_\epsilon) < \epsilon$$

and, for any $j \in \mathbb{N}_{\geq i(\epsilon)}$, $p_j|_{F_\epsilon} : F_\epsilon \rightarrow D_j$ is injective and

$$(C.11) \quad \gamma_j(D_j \setminus p_j(F_\epsilon)) < \epsilon.$$

Indeed, taking this lemma for granted, we have, for any $\epsilon > 0$ and any $j \in \mathbb{N}_{\geq i(\epsilon)}$,

$$(C.12) \quad \|\gamma_j - 1_{p_j(F_\epsilon)} \gamma_j\| < \epsilon.$$

Moreover, for any x in the finite set F_ϵ ,

$$\lim_{j \rightarrow +\infty} \gamma_j(p_j\{x\}) - \mu(\{x\}) = \lim_{j \rightarrow +\infty} \gamma_j(p_j\{x\}) - \gamma(x) = 0.$$

Therefore

$$(C.13) \quad \lim_{j \rightarrow +\infty} \|1_{p_j(F_\epsilon)} \gamma_j - p_{j*}(1_{F_\epsilon} \mu)\| = 0.$$

Finally,

$$(C.14) \quad \|1_{F_\epsilon} \mu - \mu\| = \mu(\mathcal{C} \setminus F_\epsilon) \leq \epsilon.$$

For any $(i, j) \in \mathbb{N}^2$ such that $i \leq j$, we have:

$$\begin{aligned} \|p_{ij*} \gamma_j - p_{i*} \mu\| &\leq \|p_{ij*}(\gamma_j - 1_{p_j(F_\epsilon)} \gamma_j)\| + \|p_{ij*}(1_{p_j(F_\epsilon)} \gamma_j - p_{j*}(1_{F_\epsilon} \mu))\| + \|p_{i*}(1_{F_\epsilon} \mu - \mu)\| \\ &\leq \|\gamma_j - 1_{p_j(F_\epsilon)} \gamma_j\| + \|1_{p_j(F_\epsilon)} \gamma_j - p_{j*}(1_{F_\epsilon} \mu)\| + \|1_{F_\epsilon} \mu - \mu\| \end{aligned}$$

From (C.12-C.14), we deduce that, when j goes to $+\infty$, the upper limit of the last sum is $\leq 2\epsilon$. As $\epsilon \in \mathbb{R}_+^*$ is arbitrary, this shows that

$$\lim_{j \rightarrow +\infty} \|p_{ij*} \gamma_j - p_{i*} \mu\| = 0. \quad \square$$

Proof of Lemma C.2.4. We first choose a finite subset F_ϵ of \mathcal{C} so large that

$$(C.15) \quad \sum_{x \in \mathcal{C} \setminus F_\epsilon} \gamma(x) \leq \epsilon/2.$$

Then, since F_ϵ is finite, if the integer j is large enough — say $j \geq i(\epsilon)$ — the map $p_j|_{F_\epsilon} : F_\epsilon \rightarrow D_j$ is injective and

$$(C.16) \quad \sum_{x \in F_\epsilon} |\gamma_j(\{p_j(x)\}) - \gamma(x)| \leq \epsilon/4.$$

According to (C.9), we may also assume that, when $j \geq i(\epsilon)$,

$$(C.17) \quad \gamma_j(D_j) \leq \sum_{x \in \mathcal{C}} \gamma(x) + \epsilon/8.$$

Then the estimate (C.10) follows from (C.15). To establish (C.11), write

$$\gamma_j(D_j \setminus p_j(F_\epsilon)) = \gamma_j(D_j) - \gamma_j(p_j(F_\epsilon))$$

and observe that

$$\gamma_j(p_j(F_\epsilon)) \geq \sum_{x \in F_\epsilon} \gamma(x) - \sum_{x \in F_\epsilon} |\gamma_j(\{p_j(x)\}) - \gamma(x)| = \sum_{x \in C} \gamma(x) - \sum_{x \in C \setminus F_\epsilon} \gamma(x) - \sum_{x \in F_\epsilon} |\gamma_j(\{p_j(x)\}) - \gamma(x)|.$$

Together with the estimates (C.15-C.16), this shows that

$$\gamma_j(D_j \setminus p_j(F_\epsilon)) \leq \epsilon/2 + \epsilon/4 + \epsilon/8. \quad \square$$

REFERENCES

- [Ban31] S. Banach. Über metrische Gruppen. *Stud. Math.*, 3:101–113, 1931.
- [Ban93] W. Banaszczyk. New bounds in some transference theorems in the geometry of numbers. *Math. Ann.*, 296(4):625–635, 1993.
- [BGS94] J.-B. Bost, H. Gillet, and C. Soulé. Heights of projective varieties and positive Green forms. *J. Amer. Math. Soc.*, 7(4):903–1027, 1994.
- [Bil99] P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons, Inc., New York, second edition, 1999.
- [BK10] J.-B. Bost and K. Künnemann. Hermitian vector bundles and extension groups on arithmetic schemes. I. Geometry of numbers. *Adv. Math.*, 223(3):987–1106, 2010.
- [Bol72] L. Boltzmann. Weitere Studien über Wärmegleichgewicht unter Gasmolekülen. *Wien. Ber.*, 66:275–370, 1872.
- [Bol77] L. Boltzmann. Über die Beziehungen zwischen dem zweiten Hauptsatz der Wärmetheorie und der Wahrscheinlichkeitsrechnung respektive den Sätzen über das Wärmegleichgewicht. *Wien. Ber.*, 76:373–435, 1877.
- [Bos96] J.-B. Bost. Périodes et isogénies des variétés abéliennes sur les corps de nombres (d’après D. Masser et G. Wüstholz). Séminaire Bourbaki 1994/95, Exp. no. 795. *Astérisque*, 237:115–161, 1996.
- [Bos01] J.-B. Bost. Algebraic leaves of algebraic foliations over number fields. *Publ. Math. I.H.E.S.*, 93:161–221, 2001.
- [Bou65] N. Bourbaki. *Éléments de mathématique. Fasc. XXXI. Algèbre commutative. Chapitre 7: Diviseurs*. Actualités Scientifiques et Industrielles, No. 1314. Hermann, Paris, 1965.
- [Bou69] N. Bourbaki. *Éléments de mathématique. Fasc. XXXV. Livre VI: Intégration. Chapitre IX: Intégration sur les espaces topologiques séparés*. Actualités Scientifiques et Industrielles, No. 1343. Hermann, Paris, 1969.
- [Bou74] N. Bourbaki. *Éléments de mathématique. Topologie générale. Chap. 5 à 10. Nouvelle édition*. Hermann, Paris, 1974.
- [Che52] H. Chernoff. A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations. *Ann. Math. Statistics*, 23:493–507, 1952.
- [CP11] R. Cerf and P. Petit. A short proof of Cramér’s theorem in \mathbb{R} . *Amer. Math. Monthly*, 118(10):925–931, 2011.
- [Cra38] H. Cramér. Sur un nouveau théorème-limite de la théorie des probabilités. In *Conférences internationales de sciences mathématiques (Université de Genève 1937). Théorie des probabilités. III: Les sommes et les fonctions de variables aléatoires*, volume 736 of *Actualités scientifiques et industrielles*, pages 5–23. Hermann, Paris, 1938.
- [CS93] J. H. Conway and N. J. A. Sloane. *Sphere packings, lattices and groups*, volume 290 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, New York, second edition, 1993. With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov.
- [Die50] J. Dieudonné. Matrices semi-finies et espaces localement linéairement compacts. *J. Reine Angew. Math.*, 188:162–166, 1950.
- [Ell85] R. S. Ellis. *Entropy, large deviations, and statistical mechanics*, volume 271 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, New York, 1985.
- [Eno64] E. E. Enochs. A note on reflexive modules. *Pacific J. Math.*, 14:879–881, 1964.
- [Fek23] M. Fekete. Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten. *Math. Z.*, 17(1):228–249, 1923.
- [Gas99] C. Gasbarri. Hermitian vector bundles of rank two and adjoint systems on arithmetic surfaces. *Topology*, 38(6):1161–1174, 1999.
- [Geo03] H.-O. Georgii. Probabilistic aspects of entropy. In *Entropy*, Princeton Ser. Appl. Math., pages 37–54. Princeton Univ. Press, Princeton, NJ, 2003.
- [Gib60] J. W. Gibbs. *Elementary principles in statistical mechanics: developed with especial reference to the rational foundation of thermodynamics*. Dover publications, Inc., New York, 1960. Republication of the work originally published by Yale University Press in 1902.
- [GMS91] H. Gillet, B. Mazur, and C. Soulé. A note on a classical theorem of Blichfeldt. *Bull. London Math. Soc.*, 23(2):131–132, 1991.

- [Gro01] R. P. Groenewegen. An arithmetic analogue of Clifford's theorem. *J. Théor. Nombres Bordeaux*, 13(1):143–156, 2001. 21st Journées Arithmétiques (Rome, 2001).
- [GS91] H. Gillet and C. Soulé. On the number of lattice points in convex symmetric bodies and their duals. *Israel J. Math.*, 74(2-3):347–357, 1991.
- [GS09] H. Gillet and C. Soulé. Erratum for: “On the number of lattice points in convex symmetric bodies and their duals”, *Israel Journal of Mathematics* **74** (1991), 347–357. *Israel J. Math.*, 171:443–444, 2009.
- [Hec17] E. Hecke. Über die Zetafunktion beliebiger algebraischer Zahlkörper. *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl.*, 1917:77–89, 1917.
- [Hec19] E. Hecke. Reziprozitätsgesetz und Gauß'sche Summen in quadratischen Zahlkörpern. *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl.*, 1919:265–278, 1919.
- [Hec23] E. Hecke. *Vorlesungen über die Theorie der algebraischen Zahlen*. Akademische Verlagsgesellschaft, Leipzig, 1923.
- [Hil00] D. Hilbert. Mathematische Probleme. Vortrag, gehalten auf dem internationalen Mathematiker-Congress zu Paris 1900. *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl.*, 1900:253–297, 1900.
- [Kap52] I. Kaplansky. Modules over Dedekind rings and valuation rings. *Trans. Amer. Math. Soc.*, 72:327–340, 1952.
- [Khi49] A. I. Khinchin. *Mathematical Foundations of Statistical Mechanics*. Dover Publications, Inc., New York, N. Y., 1949. Translated by G. Gamow.
- [Kol33] A. Kolmogoroff. *Grundbegriffe der Wahrscheinlichkeitsrechnung*, volume 3 of *Ergebnisse der Math. und ihrer Grenzgebiete. 2*. Julius Springer, Berlin, 1933.
- [Köt49] G. Köthe. Eine axiomatische Kennzeichnung der linearen Räume vom Typus ω . *Math. Ann.*, 120:634–649, 1949.
- [Kul97] S. Kullback. *Information theory and statistics*. Dover Publications, Inc., Mineola, NY, 1997. Reprint of the second (1968) edition.
- [Lan73] O. E. Lanford. Entropy and equilibrium states in classical statistical mechanics. In A. Lenard, editor, *Statistical mechanics and mathematical problems (Battelle Seattle 1971 Rencontres)*, pages 1–113. Lecture Notes in Physics, Vol. 20. Springer-Verlag, Berlin Heidelberg, 1973.
- [Lef42] S. Lefschetz. *Algebraic Topology*, volume 27 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, New York, 1942.
- [Man85] Yu. I. Manin. New dimensions in geometry. In *Workshop Bonn 1984 (Bonn, 1984)*, volume 1111 of *Lecture Notes in Math.*, pages 59–101. Springer, Berlin, 1985.
- [MO90] J. E. Mazo and A. M. Odlyzko. Lattice points in high-dimensional spheres. *Monatsh. Math.*, 110(1):47–61, 1990.
- [Mor95] M. Morishita. Integral representations of unramified Galois groups and matrix divisors over number fields. *Osaka J. Math.*, 32(3):565–576, 1995.
- [MR58] A. M. Macbeath and C. A. Rogers. Siegel's mean value theorem in the geometry of numbers. *Proc. Cambridge Philos. Soc.*, 54:139–151, 1958.
- [MR07] D. Micciancio and O. Regev. Worst-case to average-case reductions based on Gaussian measures. *SIAM J. Comput.*, 37(1):267–302, 2007.
- [Neu92] J. Neukirch. *Algebraische Zahlentheorie*. Berlin etc.: Springer-Verlag, 1992.
- [Roe93] D. Roessler. The Riemann-Roch theorem for arithmetic curves. Diplomarbeit, ETH Zürich, 1993.
- [Sch31] F. K. Schmidt. Analytische Zahlentheorie in Körpern der Charakteristik p . *Math. Z.*, 33:1–32, 1931.
- [Sch62] E. Schrödinger. *Statistical thermodynamics*. A course of seminar lectures delivered in January-March 1944, at the School of Theoretical Physics, Dublin Institute for Advanced Studies. Second edition, reprinted. Cambridge University Press, New York, 1962.
- [Sch73] L. Schwartz. *Radon measures on arbitrary topological spaces and cylindrical measures*, volume 6 of *Tata Institute of Fundamental Research Studies in Mathematics*. Oxford University Press, London, 1973.
- [Sie45] C. L. Siegel. A mean value theorem in geometry of numbers. *Ann. of Math. (2)*, 46:340–347, 1945.
- [Sou97] C. Soulé. Hermitian vector bundles on arithmetic varieties. In *Algebraic geometry—Santa Cruz 1995*, pages 383–419. Amer. Math. Soc., Providence, RI, 1997.
- [Spe50] E. Specker. Additive Gruppen von Folgen ganzer Zahlen. *Portugaliae Math.*, 9:131–140, 1950.
- [Str11] D. W. Stroock. *Probability theory. An analytic view. 2nd ed.* Cambridge: Cambridge University Press, 2nd ed. edition, 2011.
- [Szp85] L. Szpiro. Degrés, intersections, hauteurs. *Astérisque*, (127):11–28, 1985.
- [Tat67] J. T. Tate. Fourier analysis in number fields, and Hecke's zeta-functions. In *Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965)*, pages 305–347. Thompson, Washington, D.C., 1967. Unaltered reproduction of Tate's doctoral thesis (Princeton, May 1950).
- [TLX14] C. Tian, M. Liu, and G. Xu. Measure inequalities and the transference theorem in the geometry of numbers. *Proc. Amer. Math. Soc.*, 142(1):47–57, 2014.
- [Toe09] O. Toeplitz. Über die Auflösung unendlichvieler linearer Gleichungen mit unendlichvielen Unbekannten. *Palermo Rend.*, 28:88–96, 1909.

- [vdGS00] G. van der Geer and R. Schoof. Effectivity of Arakelov divisors and the theta divisor of a number field. *Selecta Math. (N.S.)*, 6(4):377–398, 2000.
- [Wei39] A. Weil. Sur l’analogie entre les corps de nombres algébriques et les corps de fonctions algébriques. *Rev. Sci.*, 77:104–106, 1939.
- [Wei46] A. Weil. Sur quelques résultats de Siegel. *Summa Brasil. Math.*, 1:21–39, 1946.
- [Wei82] A. Weil. *Adeles and algebraic groups*, volume 23 of *Progress in Mathematics*. Birkhäuser, Boston, Mass., 1982. (I.A.S. Lectures 1959-1960). With appendices by M. Demazure and Takashi Ono.

J.-B. BOST, DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ PARIS-SUD, BÂTIMENT 425, 91405 ORSAY CEDEX, FRANCE

E-mail address: `jean-benoit.bost@math.u-psud.fr`